

# FAIR AND PROFITABLE BILATERAL PRICES UNDER FUNDING COSTS AND COLLATERALIZATION

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## Abstract

Bielecki and Rutkowski [2] introduced and studied a generic nonlinear market model, which includes several risky assets, multiple funding accounts and margin accounts. In this paper, we examine the pricing and hedging of contract both from the perspective of the hedger and the counterparty with arbitrary initial endowments. We derive inequalities for unilateral prices and we give the range for either fair bilateral prices or bilaterally profitable prices. We also study the monotonicity of a unilateral price with respect to the initial endowment. Our study hinges on results for BSDE driven by continuous martingales obtained in [14], but we also derive the pricing PDEs for path-independent contingent claims of European style in a Markovian framework.

**Keywords:** hedging, fair prices, funding costs, margin agreement, BSDE, PDE

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# 1 Introduction

In Bielecki and Rutkowski [2], the authors introduced a generic nonlinear trading model for bilateral collateralized contracts, which includes several risky assets, multiple funding accounts, as well as the margin account. For related recent studies by other authors, see also [3, 4, 5, 7, 8, 16, 19]. Using a suitable version of the no-arbitrage argument, they first discussed the hedger's fair price for a contract in the market model without collateralization (see Section 3.2 in [2]). Subsequently, for a collateralized contract that can be replicated, they defined the hedger's ex-dividend price (see Section 5 in [2]). It was also shown in [2] that the theory of backward stochastic differential equations (BSDEs) is an important tool to compute the ex-dividend price (see, e.g., Propositions 5.2 and 5.4 in [2]). It is worth mentioning that all the pricing and hedging arguments in [2] are given from the viewpoint of the hedger and no attempt was made there to derive no-arbitrage bounds for unilateral prices and to examine the existence of fair bilateral prices. In the present work, we consistently examine the issue of pricing and hedging of an OTC derivative contract from the perspective of the hedger and his counterparty. Since we work within a nonlinear trading set-up, where the nonlinearity may stem from the different cash interest rates, funding costs for risky assets and collateralization, the hedger's and counterparty's price do not necessarily coincide. Therefore, our goal is to compare the hedger's and counterparty's prices and to derive the range for no-arbitrage prices. In the case of different lending and borrowing rates, which is a relatively simple instance of a nonlinear market model, the no-arbitrage price of any contingent claims must belong to an arbitrage band with the lower (resp., upper) bound given by the counterparty's (resp., the hedger's) price of the contract (see Bergman [1]). In a recent paper by Mercurio [13], the author extended the results from [1] by examining the pricing of European options in a model with different lending and borrowing interest rates and under collateralization.

As emphasized in [2], in the nonlinear setup (for instance, in a market model with different borrowing and lending interest rates), the initial endowments of the hedger and the counterparty play an important role in pricing considerations. Unlike in the classic options pricing model, which enjoys the linearity of the no-arbitrage pricing rule, it is no longer true that it suffices to consider the case where the initial endowments are null. This is due to the fact that, for instance, the hedger's ex-dividend price may depend on his initial endowment, in general (see Proposition 5.2 in [2]). Note in this regard that the results established in [1] and [13] only cover the case when the initial endowments of the hedger and the counterparty are null. In this paper, one of our main goals is to examine how the initial endowment of each party affects his unilateral price. For the sake of concreteness, we consider the model with partial netting and collateralization which was introduced in [2]. A similar analysis was also done for the model previously studied by Bergman [1], but with non-zero initial endowments of counterparties (see Nie and Rutkowski [15]). It is clear that the method developed in these papers can be applied to other set-ups.

This work is organized as follows. In Section 2, we give a brief overview the set-up studied in [2] and we describe the main model considered in the foregoing sections, dubbed the market model with partial netting. In Section 3, we present definitions of no-arbitrage and fair prices, as introduced in [2]. Some preliminary results from [2] are extended to the case of a collateralized contract with an exogenous margin account and we introduce and discuss the concepts of *fair bilateral prices* and *bilaterally profitable prices*. In Section 4, we show that the pricing and hedging problems for both parties can be represented by solutions of some BSDEs and we establish the existence and uniqueness results for these BSDEs. Although the BSDEs are well known to be a convenient tool to deal with prices and hedging strategies (see, e.g., [7, 8, 11]), we stress that the BSDEs studied in this work are formally derived using no-arbitrage arguments under a judiciously chosen martingale measure, whereas in some other papers on funding costs the existence of a 'risk-neutral probability' is postulated a priori, rather than formally justified. Section 5, which is the main part of this work, deals with the properties of unilateral prices. Under alternative assumptions on initial endowments of both parties, we establish several inequalities for unilateral prices, which in turn allow us to obtain the ranges for fair bilateral prices or bilaterally profitable prices. We also examine the monotonicity of prices with respect to initial endowment and we present the PDE approach within a Markovian framework. Lengthy proofs of some results are gathered in the appendix.

## 2 Trading under Funding Costs and Collateralization

Let us first recall the following setting of [2] for the market models. Throughout the paper, we fix a finite trading horizon date  $T > 0$  for our model of the financial market. Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$  models the flow of information available to all traders. For convenience, we assume that the initial  $\sigma$ -field  $\mathcal{G}_0$  is trivial. Moreover, all processes introduced in what follows are implicitly assumed to be  $\mathbb{G}$ -adapted and any semimartingale is assumed to be càdlàg.

**Risky assets.** For  $i = 1, 2, \dots, d$ , we denote by  $S^i$  the *ex-dividend price* of the  $i$ th risky asset with the *cumulative dividend stream*  $A^i$ .  $S^i$  is aimed to represent the price of any traded security, such as, stock, stock option, interest rate swap, currency option, cross-currency swap, CDS, CDO, etc.

**Cash accounts.** The riskless *lending* (resp., *borrowing*) *cash account*  $B^l$  (resp.,  $B^b$ ) is used for unsecured lending (resp., borrowing) of cash. When the borrowing and lending cash rates are equal, we denote the *cash account* simply by  $B^0$ .

**Funding accounts.** We denote by  $B^{i,l}$  (resp.,  $B^{i,b}$ ) the *lending* (resp., *borrowing*) *funding account* associated with the  $i$ th risky asset. In case when borrowing and lending rates are equal, we simply denote by  $B^i$  the *funding account* for the  $i$ th risky asset. Unless explicitly stated otherwise, we work under the assumption: long and short funding rates for each risky asset  $S^i$  are identical, that is,  $B^{i,l} = B^{i,b} = B^i$  for  $i = 1, 2, \dots, d$ .

**Assumption 2.1** The price processes of *primary assets* are assumed to satisfy:

- (i) For each  $i = 1, 2, \dots, d$ , the price  $S^i$  is semimartingale and the cumulative dividend stream  $A^i$  is finite variation process with  $A_0^i = 0$ .
- (ii) The riskless account  $B^l$ ,  $B^b$  and  $B^i$  are strictly positive and continuous finite variation processes with  $B_0^l = B_0^b = B_0^i = 1$ , for  $i = 1, 2, \dots, d$ .

For a *bilateral financial contract*, or simply a *contract*, we mean an arbitrary càdlàg process  $A$  of finite variation. The process  $A$  is aimed to represent the *cumulative cash flows* of a given contract from time 0 till its maturity date  $T$ . By convention, we set  $A_{0-} = 0$ .

The process  $A$  is assumed to model all cash flows of a given contract, which are either paid out from the wealth or added to the wealth, as seen from the perspective of the *hedger* (recall that the other party is referred to as the *counterparty*). Note that the process  $A$  includes the initial cash flow  $A_0$  of a contract at its inception date  $t_0 = 0$ . For instance, if a contract has the initial *price*  $p$  and stipulates that the hedger will receive cash flows  $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_k$  at times  $t_1, t_2, \dots, t_k \in (0, T]$ , then we set  $A_0 = p$  so that

$$A_t = p + \sum_{l=1}^k \mathbf{1}_{[t_l, T]}(t) \bar{A}_l.$$

The symbol  $p$  is frequently used to emphasize that all future cash flows  $\bar{A}_l$  for  $l = 1, 2, \dots, k$  are explicitly specified by the contract's covenants, but the initial cash flow  $A_0$  is yet to be formally defined and evaluated. Valuation of a contract  $A$  means, in particular, searching for the range of *fair values*  $p$  at time 0 from the viewpoint of either the hedger or the counterparty. Although the valuation paradigm will be the same for the two parties, due either to the asymmetry in their trading costs and opportunities, or the non-linearity of the wealth dynamics, they will typically obtain different sets of fair prices for  $A$ . This is the main objective of our current work.

### 2.1 Collateralization

In this paper, we examine the situation when the hedger and the counterparty enter a contract and either receive or post collateral with the value formally represented by a stochastic process  $C$ , which is assumed to be a semimartingale (or, at least, a càdlàg process). The process  $C$  is called the *margin account* or the *collateral amount*. Let

$$C_t = C_t \mathbf{1}_{\{C_t \geq 0\}} + C_t \mathbf{1}_{\{C_t < 0\}} = C_t^+ - C_t^-. \quad (2.1)$$

By convention,  $C_t^+$  is the cash value of collateral received at time  $t$  by the hedger, whereas  $C_t^-$  represents the cash value of collateral posted by him. For simplicity of presentation, it is postulated throughout that only cash collateral may be delivered or received (for other conventions, see [2]). We also make the following natural assumption regarding the state of the margin account at the contract's maturity date.

**Assumption 2.2** The  $\mathbb{G}$ -adapted collateral amount process  $C$  satisfies  $C_T = 0$ .

The equality  $C_T = 0$  is a convenient way of ensuring that any collateral amount posted is returned in full to its owner when a contract matures, provided that the default event does not occur at  $T$ . Of course, if the default event is also modeled, then one needs to specify the closeout payoff.

Let us first make some comments from the perspective of the hedger regarding the crucial features of the margin accounts. The current financial practice typically requires the collateral amounts to be held in *segregated* margin accounts, so that the hedger, when he is a collateral taker, cannot make use of the collateral amount for trading. Another collateral convention encountered in practice is *rehypothecation*, which refers to the situation where a bank is allowed to reuse the collateral pledged by its counterparties as collateral for its own borrowing. Note that if the hedger is a collateral giver, then a particular convention regarding segregation or rehypothecation is immaterial for the wealth dynamics of his portfolio.

We are in a position to introduce trading strategies based on a finite family of primary assets.

**Remark 2.1** For simplicity, we discuss from the point view of hedger, unless explicitly stated. The similar discussions hold for the counterparty by changing  $(A, C)$  to  $(-A, -C)$ .

**Definition 2.1** A *collateralized hedger's trading strategy* is a quadruplet  $(x, \varphi, A, C)$  where a portfolio  $\varphi$ , given by

$$\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^1, \dots, \psi^{d+1}, \eta^b, \eta^l, \eta^{d+2}) \quad (2.2)$$

is composed of the *risky assets*  $S^i$ ,  $i = 1, 2, \dots, d$ , the *unsecured lending cash account*  $B^l$  the *unsecured borrowing cash account*  $B^b$ , the *funding accounts*  $B^i$ ,  $i = 1, 2, \dots, d$ , the *borrowing account*  $B^{d+1}$  for the posted cash collateral, the *collateral accounts*  $B^{c,b}$  and  $B^{c,l}$ , and the *lending account*  $B^{d+2}$  associated with the received cash collateral.

If  $B^{c,b} \neq B^{c,l}$ , for example if the hedger post the collateral, he will receives interest from the counterparty determined by  $B^{c,l}$ , i.e., the counterparty pays the hedger the interest determined by  $B^{c,l}$  not  $B^{c,b}$ . This creates the non-identical financial environment between the hedger and counterparty. We make the following standing assumption.

**Assumption 2.3** The collateral accounts  $B^{c,l}$ ,  $B^{c,b}$ ,  $B^{d+1}$ ,  $B^{d+2}$  are strictly positive, continuous processes of finite variation with  $B_0^{c,l} = B_0^{c,b} = B_0^{d+1} = B_0^{d+2} = 1$ .

**Remark 2.2** The *cash collateral* is described by the following postulates:

- (i) If the hedger receives at time  $t$  the amount  $C_t^+$  as cash collateral, then he pays to the counterparty interest determined by the amount  $C_t^+$  and the account  $B^{c,b}$ . Under segregation, he receives interest determined by the amount  $C_t^+$  and the account  $B^{d+2}$  and thus  $\eta_t^{d+2} B_t^{d+2} = C_t^+$ . When rehypothecation is considered, the hedger may temporarily (that is, before the contract's maturity date or the default time, whichever comes first) utilize the cash amount  $C_t^+$  for his trading purposes, then  $\eta_t^{d+2} = 0$ .
- (ii) If the hedger posts a cash collateral at time  $t$ , then the collateral amount is borrowed from the dedicated *collateral borrowing account*  $B^{d+1}$ . He receives interest determined by the amount  $C_t^-$  and the collateral account  $B^{c,l}$ . We postulate that

$$\psi_t^{d+1} B_t^{d+1} = -C_t^-. \quad (2.3)$$

## 2.2 Self-Financing Trading Strategies

In the context of a collateralized contract, we find it convenient to introduce the following three processes:

- (i) the process  $V_t(x, \varphi, A, C)$  representing the hedger's wealth at time  $t$ ,
- (ii) the process  $V_t^p(x, \varphi, A, C)$  representing the value of hedger's portfolio at time  $t$ ,
- (iii) the *adjustment process*  $V_t^C(x, \varphi, A, C) := V_t(x, \varphi, A, C) - V_t^p(x, \varphi, A, C)$ , which is aimed to quantify the impact of the margin account on trading strategy.

**Definition 2.2** The hedger's *portfolio's value*  $V^p(x, \varphi, A, C)$  is given by

$$V_t^p(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \sum_{j=1}^{d+1} \psi_t^j B_t^j + \psi_t^l B_t^l + \psi_t^b B_t^b. \quad (2.4)$$

The hedger's *wealth*  $V(x, \varphi, A, C)$  equals

$$V_t(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \sum_{j=1}^{d+1} \psi_t^j B_t^j + \psi_t^l B_t^l + \psi_t^b B_t^b + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \eta_t^{d+2} B_t^{d+2}. \quad (2.5)$$

It is clear that the adjustment process  $V^C(x, \varphi, A, C)$  equals

$$V_t^C(x, \varphi, A, C) = \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \eta_t^{d+2} B_t^{d+2} = -C_t + \eta_t^{d+2} B_t^{d+2} \quad (2.6)$$

where  $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$  and  $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ .

The self-financing property of the hedger's strategy is defined in terms of the dynamics of the value process. Note that we use here the process  $V^p(x, \varphi, A, C)$ , and not  $V(x, \varphi, A, C)$ , to emphasize the role of  $V^p(x, \varphi, A, C)$  as the value of the hedger's portfolio of traded assets. Observe also that the equality  $V^p(x, \varphi, A, C) = V(x, \varphi, A, C)$  holds when the process  $C$  vanishes, that is,  $C = 0$ , since then  $\eta^{d+2} = 0$  as well. Let  $x$  stand for the *initial endowment* of the hedger.

**Definition 2.3** A collateralized hedger's trading strategy  $(x, \varphi, A, C)$  with  $\varphi$  given by (2.2) is *self-financing* whenever the *portfolio's value*  $V^p(x, \varphi, A, C)$ , which is given by (2.4), satisfies, for every  $t \in [0, T]$ ,

$$\begin{aligned} V_t^p(x, \varphi, A, C) = & x + \sum_{i=1}^d \int_{(0,t]} \xi_u^i d(S_u^i + A_u^i) + \sum_{j=1}^{d+1} \int_0^t \psi_u^j dB_u^j + \int_0^t \psi_u^l dB_u^l + \int_0^t \psi_u^b dB_u^b + A_t \\ & + \int_0^t \eta_u^b dB_u^{c,b} + \int_0^t \eta_u^l dB_u^{c,l} + \int_0^t \eta_u^{d+2} dB_u^{d+2} - V_t^C(x, \varphi, A, C). \end{aligned} \quad (2.7)$$

The terms  $\int_0^t \eta_u^b dB_u^{c,b}$ ,  $\int_0^t \eta_u^l dB_u^{c,l}$  and  $\int_0^t \eta_u^{d+2} dB_u^{d+2}$  represent the accrued interest generated by the margin account. The first two processes are given uniquely in terms of  $C$  since  $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$  and  $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ , whereas the last one depends on the collateral convention.

## 2.3 Market Model with Partial Netting

In this section, we consider a specific model with partial netting and collateralization with rehypothecation, which was previously studied in [2]. Besides postulating that the accounts  $B^l$  and  $B^b$  may differ, we also allow for the inequality  $B^{i,l} \neq B^{i,b}$ ,  $i = 1, 2, \dots, d$  to hold, in general. We also make the following simplifying assumption.

**Assumption 2.4** The collateral borrowing account  $B^{d+1}$  coincides with  $B^b$ .

We follow here the offsetting/netting terminology adopted in [2]. Hence by *offsetting* we mean the compensation of long and short positions either for a given risky asset or for the non-risky asset. This compensation is not relevant, unless the borrowing and lending rates are different for at least one risky asset or for the cash account. By *netting*, we mean the aggregation of long or short cash positions across various risky assets, which share some funding accounts. Obviously, several alternative models with netting can be studied, for more details, see [2].

In this paper, we focus on the case of partial netting positions across risky assets, which means that the offsetting of long/short positions for every risky asset combined with some form of netting of long/short cash positions for all risky assets that are funded from common funding accounts. More precisely, we postulate that all short cash positions in risky assets  $S^1, S^2, \dots, S^d$  are aggregated and invested in the common lending account  $B^l$ , which means that we assume that  $B^{i,l} = B^l$  for  $i = 1, \dots, d$ . This means that all positive cash flows, inclusive of proceeds from short-selling of risky assets, are transferred to the cash account  $B^l$ . By contrast, long cash positions in risky assets  $S^i$  are assumed to be funded from their respective funding accounts  $B^{i,b}$ . For brevity, the trading framework described in this subsection will be henceforth referred to as the *market model with partial netting*.

Accordingly, we consider a trading portfolio (note that  $\eta^{d+2} = 0$  in case of rehypothecation, as was explained in Remark 2.2)

$$\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$$

and the corresponding wealth process for the hedger

$$V_t(x, \varphi, A, C) = \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{i=1}^d (\xi_t^i S_t^i + \psi_t^{i,b} B_t^{i,b}) + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l}.$$

It follows that, for all  $t \in [0, T]$ ,

$$\eta_t^b = -(B_t^{c,b})^{-1} C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1} C_t^-, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+. \quad (2.8)$$

In the present set-up, the hedger's trading strategy  $(x, \varphi, A, C)$  is self-financing whenever the process  $V^p(x, \varphi, A, C)$ , which is given by

$$V_t^p(x, \varphi, A, C) = \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{i=1}^d (\xi_t^i S_t^i + \psi_t^{i,b} B_t^{i,b}), \quad (2.9)$$

satisfies

$$\begin{aligned} V_t^p(x, \varphi, A, C) = & x + \sum_{i=1}^d \int_{(0,t]} \xi_u^i d(S_u^i + A_u^i) + \sum_{j=1}^d \int_0^t \psi_u^{j,b} dB_u^{j,b} + \int_0^t \psi_u^l dB_u^l + \int_0^t \psi_u^b dB_u^b \\ & + \int_0^t \eta_u^b dB_u^{c,b} + \int_0^t \eta_u^l dB_u^{c,l} - V_t^C(x, \varphi, A, C) + A_t \end{aligned} \quad (2.10)$$

where in turn

$$V_t^C(x, \varphi, A, C) = \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} = -C_t.$$

From equations (2.8) and (2.9), we get

$$V_t^p(x, \varphi, A, C) = \psi_t^l B_t^l + \psi_t^b B_t^b - \sum_{i=1}^d (\xi_t^i S_t^i)^-.$$

Since we postulate that  $\psi_t^l \geq 0$ ,  $\psi_t^b \leq 0$  and  $\psi_t^l \psi_t^b = 0$  for all  $t \in [0, T]$ , we also obtain

$$\psi_t^l = (B_t^l)^{-1} \left( V_t^p(x, \varphi, A, C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+$$

and

$$\psi_t^b = -(B_t^b)^{-1} \left( V_t^p(x, \varphi, A, C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-.$$

Finally, the self-financing condition for the trading strategy  $(x, \varphi, A, C)$  can be represented as follows

$$\begin{aligned} dV_t^p(x, \varphi, A, C) &= \sum_{i=1}^d \xi_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1} dB_t^{i,b} + dA_t^C \\ &\quad + \left( V_t^p(x, \varphi, A, C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+ (B_t^l)^{-1} dB_t^l \\ &\quad - \left( V_t^p(x, \varphi, A, C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^- (B_t^b)^{-1} dB_t^b \end{aligned} \quad (2.11)$$

where  $A^C := A + C + F^C$  and, in view of Assumption 2.5,

$$\begin{aligned} F_t^C &:= \int_0^t \eta_u^b dB_u^{c,b} + \int_0^t \eta_u^l dB_u^{c,l} \\ &= - \int_0^t C_u^+ (B_u^{c,b})^{-1} dB_u^{c,b} + \int_0^t C_u^- (B_u^{c,l})^{-1} dB_u^{c,l} \\ &= - \int_0^t C_u (B_u^c)^{-1} dB_u^c. \end{aligned} \quad (2.12)$$

In general, one may consider the situation where the hedger and the counterparty are exposed to a different financial environment, which means that their respective hedging strategies for the same contract are based on different risky assets, cash accounts, funding accounts and collateral accounts. To make the analysis less cumbersome, we henceforth assume that the hedger and counterparty face exactly the same market conditions, but they may have different initial endowments. In particular, we make the following standing assumption.

**Assumption 2.5** The collateral accounts  $B^{c,l}$  and  $B^{c,b}$  satisfy  $B^{c,l} = B^{c,b} = B^c$ .

**Remark 2.3** Suppose that Assumption 2.5 is not postulated, so that the accounts  $B^{c,b}$  and  $B^{c,l}$  may be different and thus the hedger and the counterparty may be subject to different financial conditions with respect to the margin account. Then we define the process  $\Theta$  by setting

$$\Theta_t := (-A)_t^{-C} + A_t^C = - \int_0^t |C_u| (B_u^{c,b})^{-1} dB_u^{c,b} + \int_0^t |C_u| (B_u^{c,l})^{-1} dB_u^{c,l}.$$

Let us postulate, in addition, that the processes  $B^{c,b}$  and  $B^{c,l}$  are absolutely continuous, so that

$$dB_t^{c,b} = r_t^{c,b} B_t^{c,b} dt, \quad dB_t^{c,l} = r_t^{c,l} B_t^{c,l} dt,$$

for some non-negative processes  $r^{c,b}$  and  $r^{c,l}$  satisfying  $r^{c,l} \leq r^{c,b}$ . The additional assumption that  $r^{c,l} \leq r^{c,b}$  means that the counterparty has the advantage over the hedger in regard to the margin account. Indeed, when posting (resp., receiving) the collateral, the counterparty obtains a higher (resp., lower) interest than the hedger.

Under the assumption that  $r^{c,l} \leq r^{c,b}$ , the process  $\Theta$  is decreasing and thus  $\Theta_t \leq 0$  for all  $t \in [0, T]$ . Then, in all foregoing considerations in the paper, the process  $A^C$  should be replaced by  $A^C - \Theta$ . For example, in Lemma 3.1 or in Section 4 when we consider the counterparty's BSDE of the contract  $(A, C)$ , we should replace  $A^C$  by  $A^C - \Theta$  or, equivalently, replace  $F^C$  by  $F^C - \Theta$ . Since  $\Theta$  is a decreasing process, we claim that all the results will still hold, except for Theorem 5.4.

Let us finally mention that if  $r^{c,l} \geq r^{c,b}$ , which means that the hedger has the advantage over the counterparty in regard to the margin account, then the process  $\Theta$  is increasing and thus most results established in what follows will no longer be valid.



The following commonly standard assumption will allow us to derive more explicit formulae for the wealth dynamics and thus also to compute the so-called *generator* (or *driver*) for the associated BSDEs.

**Assumption 2.6** The riskless accounts are absolutely continuous, so that they can be represented as follows:

$$dB_t^l = r_t^l B_t^l dt, \quad dB_t^b = r_t^b B_t^b dt, \quad dB_t^{i,b} = r_t^{i,b} B_t^{i,b} dt, \quad (2.13)$$

for some  $\mathbb{G}$ -adapted processes  $r^l$ ,  $r^b$  and  $r^{i,b}$  for  $i = 1, 2, \dots, d$ . Moreover, we assume  $0 \leq r^l \leq r^b$  and  $r^l \leq r^{i,b}$  for  $i = 1, 2, \dots, d$ .

Let the processes  $\tilde{S}^{i,l,\text{cld}}$  and  $\tilde{S}^{i,b,\text{cld}}$  for  $i = 1, 2, \dots, d$  be given by the following expressions

$$\tilde{S}_t^{i,l,\text{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1} dA_u^i$$

and

$$\tilde{S}_t^{i,b,\text{cld}} := (B_t^b)^{-1} S_t^i + \int_{(0,t]} (B_u^b)^{-1} dA_u^i$$

so that

$$d\tilde{S}_t^{i,l,\text{cld}} = (B_t^l)^{-1} (dS_t^i - r_t^l S_t^i dt + dA_t^i) \quad (2.14)$$

and

$$d\tilde{S}_t^{i,b,\text{cld}} = (B_t^b)^{-1} (dS_t^i - r_t^b S_t^i dt + dA_t^i). \quad (2.15)$$

We also denote

$$A_t^{C,l} := \int_{(0,t]} (B_u^l)^{-1} dA_u^C, \quad A_t^{C,b} := \int_{(0,t]} (B_u^b)^{-1} dA_u^C.$$

In view (2.11), the following lemmas are straightforward (see also Lemma 5.1 and Remark 5.3 in [2]).

**Lemma 2.1** The discounted wealth  $Y^l := \tilde{V}^{p,l}(x, \varphi, A, C) = (B^l)^{-1} V^p(x, \varphi, A, C)$  satisfies

$$dY_t^l = \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + \tilde{f}_l(t, Y_t^l, \xi_t) dt + dA_t^{C,l}$$

where the mapping  $\tilde{f}_l : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is given by

$$\tilde{f}_l(t, y, z) := (B_t^l)^{-1} f_l(t, B_t^l y, z) - r_t^l y \quad (2.16)$$

and  $f_l : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  equals

$$f_l(t, y, z) := \sum_{i=1}^d r_t^l z^i S_t^i - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^-.$$

**Lemma 2.2** The discounted wealth  $Y^b := \tilde{V}^{p,b}(x, \varphi, A, C) = (B^b)^{-1} V^p(x, \varphi, A, C)$  satisfies

$$dY_t^b = \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,b,\text{cld}} + \tilde{f}_b(t, Y_t^b, \xi_t) dt + dA_t^{C,b}$$

where the mapping  $\tilde{f}_b : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is given by

$$\tilde{f}_b(t, y, z) := (B_t^b)^{-1} f_b(t, B_t^b y, z) - r_t^b y \quad (2.17)$$

and  $f_b : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  equals

$$f_b(t, y, z) := \sum_{i=1}^d r_t^b z^i S_t^i - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^-.$$



### 3 Arbitrage Opportunities and Ex-Dividend Prices

We consider throughout the hedger's self-financing trading strategies  $(x, \varphi, A, C)$ , as specified by Definition 2.3, where  $x$  is the hedger's initial endowment. We set  $V_T^0(x) := xB_T^l \mathbf{1}_{\{x \geq 0\}} + xB_T^b \mathbf{1}_{\{x < 0\}}$  and we define the *discounted wealth process*  $\widehat{V}(x, \varphi, A, C)$  by the following expression, for all  $t \in [0, T]$ ,

$$\widehat{V}_t(x, \varphi, A, C) := (B_t^l)^{-1} V_t(x, \varphi, A, C) \mathbf{1}_{\{x \geq 0\}} + (B_t^b)^{-1} V_t(x, \varphi, A, C) \mathbf{1}_{\{x < 0\}}.$$

#### 3.1 Netted Wealth and Arbitrage Opportunities

We first extend the results obtained in Section 3 of [2] to the case of a collateralized contract. For the financial interpretation of the *netted wealth*, the reader is referred to [2]. Let us only mention that  $A_0 = p^{A,C} \in \mathbb{R}$  stands here for a generic price of a contract at time 0, as seen from the perspective of the hedger.

**Definition 3.1** The *netted wealth*  $V^{\text{net}}(x, \varphi, A, C)$  of a trading strategy  $(x, \varphi, A, C)$  is given by  $V^{\text{net}}(x, \varphi, A, C) := V(x, \varphi, A, C) + V(0, \widetilde{\varphi}, -A, -C)$  where  $(0, \widetilde{\varphi}, -A, -C)$  is the unique self-financing strategy satisfying the following conditions:

- (i)  $V_0(0, \widetilde{\varphi}, -A) = -A_0$ ,
- (ii)  $\widetilde{\xi}_t^i = 0$  (hence  $\widetilde{\psi}_t^{i,b} = 0$  in view of (2.8)) for all  $i = 1, 2, \dots, d$  and  $t \in [0, T]$ ,
- (iii)  $\widetilde{\psi}_t^l \geq 0$ ,  $\widetilde{\psi}_t^b \leq 0$  and  $\widetilde{\psi}_t^l \widetilde{\psi}_t^b = 0$  for all  $t \in [0, T]$ .

We note that

$$V_0^{\text{net}}(x, \varphi, A, C) = V_0(x, \varphi, A, C) + V_0(0, \widetilde{\varphi}, -A, -C) = x + A_0 + C_0 - A_0 - C_0 = x,$$

so that the initial netted wealth  $V_0^{\text{net}}(x, \varphi, A, C)$  is independent of  $(A_0, C_0)$  and it simply equals the hedger's initial endowment.

**Definition 3.2** A self-financing trading strategy  $(x, \varphi, A, C)$  is *admissible for the hedger* whenever the discounted netted wealth process  $\widehat{V}^{\text{net}}(x, \varphi, A, C)$  is bounded from below by a constant.

**Definition 3.3** An admissible trading strategy  $(x, \varphi, A, C)$  is an *arbitrage opportunity for the hedger* with respect to  $(A, C)$  whenever

$$\mathbb{P}(V_T^{\text{net}}(x, \varphi, A, C) \geq V_T^0(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{\text{net}}(x, \varphi, A, C) > V_T^0(x)) > 0.$$

A market model is *arbitrage-free* for the hedger if no arbitrage opportunities for the hedger exist in regard to any contract  $(A, C)$ .

The condition that the discounted netted wealth process  $\widehat{V}^{\text{net}}(x, \varphi, A, C)$  is bounded from below by a constant is a commonly used criterion of *admissibility*, which ensures that, if the process  $\widehat{V}^{\text{net}}(x, \varphi, A, C)$  a local martingale under some equivalent probability measure, then it is also a supermartingale. It is well known that some technical assumption of this nature cannot be avoided even in the classic case of the Black and Scholes model.

**Lemma 3.1** We have  $V^{\text{net}}(x, \varphi, A, C) = V(x, \varphi, A, C) + U(A, C)$ , where the  $\mathbb{G}$ -adapted process of finite variation  $U(A, C) = U$  is the unique solution to the following equation

$$U_t = \int_0^t (B_u^l)^{-1} (U_u - C_u)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (U_u - C_u)^- dB_u^b - A_t - F_t^C \quad (3.1)$$

where  $F^C$  is defined by (2.12).

*Proof.* We set  $\tilde{\xi}^i = \tilde{\psi}^{i,b} = 0$  in (2.9) and (2.10). Then the process  $V^p := V^p(0, \tilde{\psi}^l, \tilde{\psi}^b, \eta^b, \eta^l, -A, -C)$  satisfies  $V_t^p = \tilde{\psi}_t^l B_t^l + \tilde{\psi}_t^b B_t^b$  for every  $t \in [0, T]$ . Noting that  $V^c := V^c(0, \tilde{\psi}^l, \tilde{\psi}^b, \eta^b, \eta^l, -A, -C) = C$  and recalling the definition of  $F^C$  and  $A^C$ , we obtain

$$\begin{aligned} V_t^p &= \int_0^t (B_u^l)^{-1} (V_u^p)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (V_u^p)^- dB_u^b - A_t + F_t^{-C} - V_t^c(0, \tilde{\psi}^l, \tilde{\psi}^b, \eta^b, \eta^l, -A, -C) \\ &= \int_0^t (B_u^l)^{-1} (V_u^p)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (V_u^p)^- dB_u^b - A_t - F_t^C - C_t \\ &= \int_0^t (B_u^l)^{-1} (V_u^p)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (V_u^p)^- dB_u^b - A_t^C. \end{aligned}$$

Consequently, the process  $V := V(0, \tilde{\psi}^l, \tilde{\psi}^b, \eta^b, \eta^l, -A, -C) = V^p + V^c$  satisfies

$$V_t = \int_0^t (B_u^l)^{-1} (V_u - C_u)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (V_u - C_u)^- dB_u^b - A_t - F_t^C$$

and thus the assertion of the lemma follows.  $\square$

**Assumption 3.1** There exists a probability measure  $\tilde{\mathbb{P}}^l$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,l,\text{cld}}$ ,  $i = 1, 2, \dots, d$  are  $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -local martingales.

**Proposition 3.1** Under Assumptions 2.6 and 3.1, if  $x \geq 0$ , then the market model of Section 2.3 is arbitrage-free for the hedger in regard to any contract  $(A, C)$ .

*Proof.* In view of (2.11) and the postulated inequalities:  $r^l \leq r^b$  and  $r^l \leq r^{i,b}$  for all  $i$ , the process  $V^p := V^p(x, \varphi, A, C)$  satisfies

$$\begin{aligned} dV_t^p &= \sum_{i=1}^d \xi_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\xi_t^i S_t^i)^+ dt + dA_t^C \\ &\quad + r_t^l \left( V_t^p + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+ dt - r_t^b \left( V_t^p + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^- dt \\ &\leq \sum_{i=1}^d \xi_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\xi_t^i S_t^i)^+ dt + dA_t^C \\ &\quad + r_t^l \left( V_t^p + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+ dt - r_t^l \left( V_t^p + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^- dt \\ &= r_t^l V_t^p dt + \sum_{i=1}^d \xi_t^i (dS_t^i + dA_t^i) + dA_t^C - \sum_{i=1}^d r_t^{i,b} (\xi_t^i S_t^i)^+ dt + r_t^l \sum_{i=1}^d (\xi_t^i S_t^i)^- dt \\ &\leq r_t^l V_t^p dt + \sum_{i=1}^d \xi_t^i (dS_t^i - r_t^l S_t^i dt + dA_t^i) + dA_t^C. \end{aligned}$$

Consequently, the discounted wealth  $V^{l,p} := (B^l)^{-1} V^p$  satisfies

$$dV_t^{l,p} \leq \sum_{i=1}^d \xi_t^i (B_t^l)^{-1} (dS_t^i - r_t^l S_t^i dt + dA_t^i) + (B_t^l)^{-1} dA_t^C = \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + dA_t^{C,l}.$$

Furthermore, the netted wealth is given by the following expression (see Lemma 3.1)

$$V_t^{\text{net}}(x, \varphi, A, C) = V_t(x, \varphi, A, C) + U_t(A, C) = V_t^p - C_t + U_t(A, C)$$

where the  $\mathbb{G}$ -adapted process of finite variation  $U(A, C)$  is given by (3.1). Hence the discounted netted wealth, which is given by

$$\tilde{V}_t^{l,\text{net}} := (B_t^l)^{-1} V_t^{\text{net}}(x, \varphi, A, C) = V_t^{l,p} - (B_t^l)^{-1} C_t + (B_t^l)^{-1} U_t(A, C),$$

satisfies (for brevity, we write  $U(A, C) = U$ )

$$\begin{aligned} d\tilde{V}_t^{l,\text{net}} &= dV_t^{l,p} - d((B_t^l)^{-1} C_t) + d((B_t^l)^{-1} U_t) \\ &\leq \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + (B_t^l)^{-2} (U_t - C_t)^+ dB_t^l - (B_t^l)^{-1} (B_t^b)^{-1} (U_t - C_t)^- dB_t^b \\ &\quad + U_t d(B_t^l)^{-1} + (B_t^l)^{-1} dA_t^C - (B_t^l)^{-1} dA_t - (B_t^l)^{-1} dF_t^C - d((B_t^l)^{-1} C_t) \\ &= \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + (r_t^l - r_t^b) (B_t^l)^{-1} (U_t - C_t)^- dt \end{aligned}$$

and thus

$$\tilde{V}_t^{l,\text{net}} - \tilde{V}_0^{l,\text{net}} \leq \sum_{i=1}^d \int_{(0,t]} \xi_u^i d\tilde{S}_u^{i,l,\text{cld}}. \quad (3.2)$$

First, the assumption that the process  $\tilde{V}^{l,\text{net}}$  is bounded from below, implies that the right-hand side in (3.2) is a  $(\mathbb{P}^l, \mathbb{G})$ -supermartingale, which is null at  $t = 0$ . Next,  $V_T^0(x) = B_T^l x$  (since  $x \geq 0$ ). From (3.2), we thus obtain

$$(B_T^l)^{-1} (V_T^{\text{net}}(x, \varphi, A) - V_T^0(x)) \leq \sum_{i=1}^d \int_{(0,T]} \xi_u^i d\tilde{S}_u^{i,l,\text{cld}}.$$

Since  $\tilde{\mathbb{P}}^l$  is equivalent to  $\mathbb{P}$ , we conclude that either  $V_T^{\text{net}}(x, \varphi, A, C) = V_T^0(x)$  or  $\mathbb{P}(V_T^{\text{net}}(x, \varphi, A, C) < V_T^0(x)) > 0$ . This means that an arbitrage opportunity may not arise and thus the market model with partial netting is arbitrage-free for the hedger in regard to any contract  $(A, C)$ .  $\square$

**Assumption 3.2** There exists a probability measure  $\tilde{\mathbb{P}}^b$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,b,\text{cld}}$ ,  $i = 1, 2, \dots, d$  are  $(\tilde{\mathbb{P}}^b, \mathbb{G})$ -local martingales.

**Remark 3.1** Similarly as in Remark 3.2 of [2], we observe that the statement of Proposition 3.1 is also true for  $x \leq 0$ , provided that Assumption 3.2 is valid and  $r^b \leq r^{i,b}$  for  $i = 1, 2, \dots, d$ . For the hedger, one can then show that

$$d\tilde{V}_t^{b,\text{net}} \leq \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,b,\text{cld}} + (r_t^l - r_t^b) (B_t^b)^{-1} (U_t(A, C) - C_t)^+ dt$$

and thus, using similar arguments as above, we conclude that there is no arbitrage for the hedger in regard to any contract  $(A, C)$ .

**Remark 3.2** Let Assumption 2.5 be valid. Then Definition 3.3, Proposition 3.1 and Remark 3.1 apply not only to the hedger, but also to the counterparty. Therefore, if both parties have non-negative initial endowments (resp., both have non-positive initial endowments), Assumption 3.1 (resp., Assumption 3.2) holds, and  $r^l \leq r^{i,b}$  (resp.,  $r^b \leq r^{i,b}$ ) for all  $i$ , then the model is arbitrage-free for both parties. When the initial endowments have opposite signs then if Assumptions 3.1 and 3.2 are valid and  $r^b \leq r^{i,b}$  for all  $i$ , then the model is arbitrage-free for both parties.

## 3.2 Extended Arbitrage Opportunities

Results of Section 3.1 give only a partial answer to the question whether a market model with partial netting is arbitrage-free. We will now attempt to give a deeper analysis of the arbitrage-free property for all contracts under specific assumptions on prices of risky assets. To this end, we introduce the following definition (see Remark 3.1 in [2]).

**Definition 3.4** An *extended arbitrage opportunity* with respect to the contract  $(A, C)$  for the hedger with the initial endowment  $x$  is a pair  $(\hat{x}, \hat{\varphi}, A)$  and  $(\tilde{x}, \tilde{\varphi}, -A)$  of admissible strategies such that  $x = \hat{x} + \tilde{x}$  and

$$\mathbb{P}(V_T^{\text{net}} \geq V_T^0(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{\text{net}} > V_T^0(x)) > 0$$

where the *netted wealth*  $V^{\text{net}} = V^{\text{net}}(\hat{x}, \tilde{x}, \hat{\varphi}, \tilde{\varphi}, A, C)$  is given by

$$V^{\text{net}} := V(\hat{x}, \hat{\varphi}, A, C) + V(\tilde{x}, \tilde{\varphi}, -A, -C).$$

The next result gives sufficient conditions for non-existence of extended arbitrage opportunities for the hedger.

**Proposition 3.2** Assume that there exist some  $\mathbb{G}$ -adapted processes  $\beta^i$  satisfying  $r^b \leq \beta^i \leq r^{i,b}$  and a probability measure  $\tilde{\mathbb{P}}^\beta$  equivalent to  $\mathbb{P}$  such that the auxiliary processes  $\tilde{S}^{i,\text{cld}}$ ,  $i = 1, 2, \dots, d$ , which are given by

$$d\tilde{S}_t^{i,\text{cld}} = dS_t^i + dA_t^i - \beta_t^i S_t^i dt, \quad (3.3)$$

are continuous, square-integrable,  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -martingales. Then no extended arbitrage opportunity exists for the hedger in respect of any contract  $(A, C)$  and any initial endowment  $x \in \mathbb{R}$ .

*Proof.* Note that the process  $\hat{V}^p := V^p(\hat{x}, \hat{\varphi}, A, C)$  is governed by

$$\begin{aligned} d\hat{V}_t^p &= \sum_{i=1}^d \hat{\xi}_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\hat{\xi}_t^i S_t^i)^+ dt + dA_t^C \\ &\quad + r_t^l \left( \hat{V}_t^p + \sum_{i=1}^d (\hat{\xi}_t^i S_t^i)^- \right)^+ dt - r_t^b \left( \hat{V}_t^p + \sum_{i=1}^d (\hat{\xi}_t^i S_t^i)^- \right)^- dt. \end{aligned}$$

and  $\tilde{V}^p := V^p(\tilde{x}, \tilde{\varphi}, -A, -C)$  satisfies

$$\begin{aligned} d\tilde{V}_t^p &= \sum_{i=1}^d \tilde{\xi}_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\tilde{\xi}_t^i S_t^i)^+ dt - dA_t^C \\ &\quad + r_t^l \left( \tilde{V}_t^p + \sum_{i=1}^d (\tilde{\xi}_t^i S_t^i)^- \right)^+ dt - r_t^b \left( \tilde{V}_t^p + \sum_{i=1}^d (\tilde{\xi}_t^i S_t^i)^- \right)^- dt. \end{aligned}$$

We observe that the netted wealth satisfies

$$V^{\text{net}} := V(\hat{x}, \hat{\varphi}, A, C) + V(\tilde{x}, \tilde{\varphi}, -A, -C) = \hat{V}^p - C + \tilde{V}^p + C = \hat{V}^p + \tilde{V}^p$$

and thus

$$\begin{aligned} dV_t^{\text{net}} &= \sum_{i=1}^d (\hat{\xi}_t^i + \tilde{\xi}_t^i) (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\hat{\xi}_t^i S_t^i)^+ dt - \sum_{i=1}^d r_t^{i,b} (\tilde{\xi}_t^i S_t^i)^+ dt \\ &\quad + r_t^l \left( \hat{V}_t^p + \sum_{i=1}^d (\hat{\xi}_t^i S_t^i)^- \right)^+ dt - r_t^b \left( \hat{V}_t^p + \sum_{i=1}^d (\hat{\xi}_t^i S_t^i)^- \right)^- dt \\ &\quad + r_t^l \left( \tilde{V}_t^p + \sum_{i=1}^d (\tilde{\xi}_t^i S_t^i)^- \right)^+ dt - r_t^b \left( \tilde{V}_t^p + \sum_{i=1}^d (\tilde{\xi}_t^i S_t^i)^- \right)^- dt. \end{aligned}$$

Since  $r^l \leq r^b$ , we obtain

$$\begin{aligned} dV_t^{\text{net}} &\leq \sum_{i=1}^d (\hat{\xi}_t^i + \tilde{\xi}_t^i) (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\hat{\xi}_t^i S_t^i)^+ dt - \sum_{i=1}^d r_t^{i,b} (\tilde{\xi}_t^i S_t^i)^+ dt \\ &\quad + r_t^l \left( \hat{V}_t^p + \tilde{V}_t^p + \sum_{i=1}^d (\hat{\xi}_t^i S_t^i)^- + \sum_{i=1}^d (\tilde{\xi}_t^i S_t^i)^- \right) dt \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} dV_t^{\text{net}} &\leq \sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i) (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ dt - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ dt \\ &\quad + r_t^b \left( \widehat{V}_t^p + \widetilde{V}_t^p + \sum_{i=1}^d (\widehat{\xi}_t^i S_t^i)^- + \sum_{i=1}^d (\widetilde{\xi}_t^i S_t^i)^- \right) dt. \end{aligned} \quad (3.5)$$

Using (3.4) and the equality  $V^{\text{net}} = \widehat{V}^p + \widetilde{V}^p$ , we obtain for the process  $\widetilde{V}^{l,\text{net}} := (B^l)^{-1} V^{\text{net}}$

$$\begin{aligned} d\widetilde{V}_t^{l,\text{net}} &= (B_t^l)^{-1} dV_t^{\text{net}} - r_t^l (B_t^l)^{-1} V_t^{\text{net}} dt \\ &\leq (B_t^l)^{-1} \left( \sum_{i=1}^d \widehat{\xi}_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^- dt \right) \\ &\quad + (B_t^l)^{-1} \left( \sum_{i=1}^d \widetilde{\xi}_t^i (dS_t^i + dA_t^i) - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^- dt \right) \\ &= (B_t^l)^{-1} \left( \sum_{i=1}^d \widehat{\xi}_t^i (dS_t^i + dA_t^i - \beta_t^i S_t^i dt) - \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^- dt + \sum_{i=1}^d \beta_t^i \widehat{\xi}_t^i S_t^i dt \right) \\ &\quad + (B_t^l)^{-1} \left( \sum_{i=1}^d \widetilde{\xi}_t^i (dS_t^i + dA_t^i - \beta_t^i S_t^i dt) - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^- dt + \sum_{i=1}^d \beta_t^i \widetilde{\xi}_t^i S_t^i dt \right). \end{aligned}$$

Similarly, in view (3.5), the process  $\widetilde{V}^{b,\text{net}} := (B^b)^{-1} V^{\text{net}}$  satisfies

$$\begin{aligned} d\widetilde{V}_t^{b,\text{net}} &= (B_t^b)^{-1} dV_t^{\text{net}} - r_t^b (B_t^b)^{-1} V_t^{\text{net}} dt \\ &= (B_t^b)^{-1} \left( \sum_{i=1}^d \widehat{\xi}_t^i (dS_t^i + dA_t^i - \beta_t^i S_t^i dt) - \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^b (\widehat{\xi}_t^i S_t^i)^- dt + \sum_{i=1}^d \beta_t^i \widehat{\xi}_t^i S_t^i dt \right) \\ &\quad + (B_t^b)^{-1} \left( \sum_{i=1}^d \widetilde{\xi}_t^i (dS_t^i + dA_t^i - \beta_t^i S_t^i dt) - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ dt + \sum_{i=1}^d r_t^b (\widetilde{\xi}_t^i S_t^i)^- dt + \sum_{i=1}^d \beta_t^i \widetilde{\xi}_t^i S_t^i dt \right). \end{aligned}$$

Since the process  $\beta^i$  satisfies  $r^l \leq \beta^i \leq r^{i,b}$  for every  $i = 1, 2, \dots, d$ , we obtain

$$\sum_{i=1}^d \beta_t^i \widehat{\xi}_t^i S_t^i \leq \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ - \sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^-.$$

Under the stronger condition that  $r^b \leq \beta^i \leq r^{i,b}$  is satisfied for every  $i = 1, 2, \dots, d$ , we also have that

$$\sum_{i=1}^d \beta_t^i \widehat{\xi}_t^i S_t^i \leq \sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ - \sum_{i=1}^d r_t^b (\widehat{\xi}_t^i S_t^i)^-.$$

By assumption, there exists a probability measure  $\widetilde{\mathbb{P}}^\beta$  equivalent to  $\mathbb{P}$  such that the processes  $\widetilde{S}^{i,\text{cld}}, i = 1, 2, \dots, d$  are continuous, square-integrable,  $(\widetilde{\mathbb{P}}^\beta, \mathbb{G})$ -martingales, where  $\widetilde{S}^{i,\text{cld}}$  is given by (3.3) for some  $\mathbb{G}$ -adapted processes  $\beta^i$  satisfying  $r^b \leq \beta^i \leq r^{i,b}$ . Then

$$\widetilde{V}_t^{l,\text{net}} - \widetilde{V}_0^{l,\text{net}} \leq \sum_{i=1}^d \int_{(0,t]} (\widehat{\xi}_u^i + \widetilde{\xi}_u^i) d\widetilde{S}_u^{i,\text{cld}} \quad (3.6)$$

and

$$\widetilde{V}_t^{b,\text{net}} - \widetilde{V}_0^{b,\text{net}} \leq \sum_{i=1}^d \int_{(0,t]} (\widehat{\xi}_u^i + \widetilde{\xi}_u^i) d\widetilde{S}_u^{i,\text{cld}}. \quad (3.7)$$

Using standard arguments (see the proof of Proposition 3.1), we deduce that the market model with partial netting is arbitrage-free for the hedger in respect of any contract  $(A, C)$  and any initial endowment  $x \in \mathbb{R}$ .  $\square$

Let us now discuss various alternative martingale conditions, which were introduced to analyze the non-existence of (extended) arbitrage opportunities in the present set-up. First, it is easy to see that Proposition 3.2 furnishes sufficient conditions ensuring that the market model with partial netting is arbitrage-free with respect to any contract for both parties with arbitrary initial endowments  $x_1, x_2 \in \mathbb{R}$ . This motivates the introduction of Assumptions 5.1 and 5.2 in Section 5.2. It is fair to acknowledge that the condition  $r^b \leq \beta^i \leq r^{i,b}$  is restrictive and thus this result is not fully satisfactory. However, we argue below that the condition  $r^b \leq \beta^i \leq r^{i,b}$  is needed in the abstract set-up where ‘risky’ assets are left unspecified, so their prices may in fact be modeled through continuous processes of finite variation.

It is also worth noting that the condition  $r^b \leq r^{i,b}$  in Remark 3.1 was not due to the fact that we considered there the case when  $x \leq 0$ , but rather to the choice of  $B^b$  as a discount factor. Specifically, we decided to search for a sufficient condition for arbitrage-free property in terms of a martingale measure for processes  $\tilde{S}^{i,b,\text{cld}}$ . Since we do not make any a priori assumptions about the price processes for risky assets, it may happen that, for instance,  $S^1 = S_0^1 B^b$  and  $A^1 = 0$ . Of course, a martingale measure for the process  $\tilde{S}^{1,b,\text{cld}}$  exists, but the sub-model  $(B^l, B^b, B^{1,b}, S^1)$  is not arbitrage-free unless  $r^b \leq r^{1,b}$ . Indeed, in the present set-up, the rate  $r^{1,b}$  (resp.  $r^b$ ) can be seen as a borrowing (resp., lending) rate, since the non-risky return  $r^b$  can be generated by the hedger by purchasing the stock  $S^1$ . This argument shows that the inequality  $r^b \leq r^{i,b}$  is necessary to avoid arbitrage if we do not make any other assumptions about risky asset except for postulating the existence of a martingale measure for  $\tilde{S}^{1,b,\text{cld}}$  (by contrast, if the stock price  $S^1$  is given, say, by the Black and Scholes model then there is no need to postulate that  $r^b \leq r^{i,b}$  since any investment in  $S^1$  is risky).

The condition that a martingale measure for  $\tilde{S}^{1,b,\text{cld}}$  exists is, in some sense, weaker than the postulate that a martingale measure for  $\tilde{S}^{1,l,\text{cld}}$  exists. Indeed, in the latter case, when the asset price is of finite variation, it equals to  $S_0 B^l$  (rather than  $S_0 B^b$ ) and thus one could conjecture that the condition  $r^l \leq r^{1,b}$  is sufficient to preclude arbitrage in the sub-model  $(B^l, B^b, B^{1,b}, S^1)$ . This is indeed true when  $x \geq 0$ , but when  $x < 0$  and  $r^l < r^b$ , there still exists an arbitrage opportunity, since the hedger may sell stock and reduce interest payments on his debt.

Finally, one could postulate that the process  $B^{i,b}$  could be chosen as a discount factor for the  $i$ th risky asset. In that case, to preclude an arbitrage opportunity of the same kind as above when  $x < 0$ , one would need to postulate that  $r^{i,b} \geq r^b$ .

In our opinion, the condition that a martingale measure for  $\tilde{S}^{i,l,\text{cld}}$  exists is more natural but, as was explained above, it is not a sufficient condition for no-arbitrage if a ‘risky’ asset may in fact be non-risky. Of course, in a non-trivial model where the prices of risky assets have non-vanishing volatilities, the above-mentioned martingale conditions will be equivalent, under mild technical assumptions, and conditions  $r^l \leq r^b$  and  $r^l \leq r^{i,b}$  that underpin Proposition 3.1 should suffice to ensure that a model is arbitrage-free for both parties with arbitrary initial endowments.

### 3.3 Fair and Profitable Bilateral Prices

Our next goal is to describe the range of arbitrage prices of a contract with cash flows  $A$  and collateral  $C$ . It is rather clear from the next definition that a *hedger’s fair price* may depend on the hedger’s initial endowment  $x$  and it may fail to be unique, in general.

**Definition 3.5** We say that a real number  $p^{A,C} = A_0$  is a *hedger’s fair price* for  $(A, C)$  at time 0 whenever for any self-financing trading strategy  $(x, \varphi, A, C)$ , such that the discounted wealth process  $\hat{V}(x, \varphi, A, C)$  is bounded from below, we have that

$$\mathbb{P}(V_T(x, \varphi, A, C) = V_T^0(x)) = 1 \quad \text{or} \quad \mathbb{P}(V_T(x, \varphi, A, C) < V_T^0(x)) > 0. \quad (3.8)$$

One may observe that the two conditions in Definition 3.5 are analogous to conditions of Definition 3.3, although they have different financial meaning. Recall that Definition 3.3 deals with a possibility of offsetting a dynamically hedged contract  $(A, C)$  with an arbitrary market price by an unhedged contract  $(-A, -C)$ , whereas Definition 3.5 is concerned with finding a unilateral fair price for  $(A, C)$  from the perspective of the hedger. For a more detailed discussion, the interested reader may consult [2].

Let us recall the generic definition of replication of a contract on  $[t, T]$  (see Definition 5.1 in [2]).

**Definition 3.6** For a fixed  $t \in [0, T]$ , a self-financing trading strategy  $(V_t^0(x) + p_t^{A,C}, \varphi, A - A_t, C)$ , where  $p_t^{A,C}$  is a  $\mathcal{G}_t$ -measurable random variable, is said to *replicate the collateralized contract*  $(A, C)$  on  $[t, T]$  whenever  $V_T(V_t^0(x) + p_t^{A,C}, \varphi, A - A_t, C) = V_T^0(x)$ .

We henceforth assume that the initial endowment of the hedger (resp., the counterparty) is  $x_1$  (resp.,  $x_2$ ) where  $x_1, x_2 \in \mathbb{R}$ . We consider the situation when the hedger with the initial endowment  $x_1$  at time 0 enters the contract  $A$  at time  $t$  and the contract can be replicated by the hedger.

**Definition 3.7** Any  $\mathcal{G}_t$ -measurable random variable for which a replicating strategy for  $A$  over  $[t, T]$  exists is called the *hedger's ex-dividend price* at time  $t$  for the contract  $(A, C)$  and it is denoted by  $P_t^h(x_1, A, C)$ , so that for some  $\varphi$  replicating  $(A, C)$

$$V_T(V_t^0(x_1) + P_t^h(x_1, A, C), \varphi, A - A_t, C) = V_T^0(x_1).$$

**Definition 3.8** For an arbitrary level  $x_2$  of the counterparty's initial endowment and a strategy  $\varphi$  replicating  $(-A, -C)$ , the *counterparty's ex-dividend price*  $P_t^c(x_2, -A, -C)$  at time  $t$  for the contract  $(-A, -C)$  is given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \varphi, -A + A_t, -C) = V_T^0(x_2).$$

It is clear that in Definitions 3.7 and 3.8, we deal with unilateral prices, as evaluated by the hedger and the counterparty, respectively. Note that if  $x_1 = x_2 = x$ , then  $P_t^h(x, A, C) = p_t^{A,C}$  and  $P_t^c(x, -A, -C) = -p_t^{A,C}$ . Due to this convention, the equality  $P_t^h(x_1, A, C) = P_t^c(x_1, -A, -C)$  will be satisfied when Definitions 3.7 and 3.8 applied to a standard market model with a single cash account in which the prices are known to be independent of initial endowments  $x_1$  and  $x_2$ . The next definition is consistent with this convention. Note that Definition 3.9 is based on an implicit assumption that prices are uniquely defined; we address this important issue in the foregoing section.

**Definition 3.9** The hedger is willing to *sell* (resp., to *buy*) a contract  $(A, C)$  if  $P_t^h(x_1, A, C) \geq 0$  (resp.,  $P_t^h(x_1, A, C) \leq 0$ ). The counterparty is willing to *sell* (resp., to *buy*) a contract  $(-A, -C)$  if  $P_t^c(x_2, -A, -C) \leq 0$  (resp.,  $P_t^c(x_2, -A, -C) \geq 0$ ).

Since we place ourselves in a nonlinear framework, a natural asymmetry arises between the hedger and his counterparty, so that the price discrepancy may occur, meaning that it may happen that  $P_t^h(x_1, A, C) \neq P_t^c(x_2, -A, -C)$ . However, it is expected that the two prices will typically yield a no-arbitrage range determined by the (higher) seller's price and the (lower) buyer's price, though it may also happen that both parties are willing to be sellers (or both are willing to be buyers) of a given contract. In addition, since a positive excess cash generated by one contract may be offset (partly or totally) by a negative excess cash associated with another contract, we expect that the seller's (resp., buyer's) price for the combination of two contracts should be lower (resp., higher) than the sum of the seller's (resp., buyer's) prices of individual contracts.

**Example 3.1** Let us consider a contract  $(A, C)$  with  $C = 0$  and  $A_t = p \mathbf{1}_{[0, T]}(t) + X \mathbf{1}_{[T]}(t)$ . If  $X = -(S_T^i - K)^+$ , then we deal with a European call option written by the hedger. A natural guess is that the prices  $P_0^h(x_1, A, C)$  and  $P_0^c(x_2, -A, -C)$  should be positive. Similarly, if  $X = (S_T^i - K)^+$ , that



the counterparty is the option's writer, it is natural to expect that  $P_0^h(x_1, A, C)$  and  $P_0^c(x_2, -A, -C)$  should be negative. Furthermore, if  $C = 0$  and  $A_t = p \mathbf{1}_{[0, T]}(t) - (S_T^i - K)^+ \mathbf{1}_{[T]}(t)$ , then we guess that the price  $P_0^h(x_1, A, C)$  should be independent of  $x_1$ , provided that  $x_1 \geq 0$ . Indeed, as a consequence of the last constraint in (2.8), the hedger cannot use his initial endowment to buy shares for the purpose of hedging. In view of this constraint, the postulated model does not cover the standard case of different borrowing and lending rates when  $r^{i,b} = r^b > r^l$  and trading is assumed to be unrestricted, so that the hedger's initial endowment can be used for hedging.

In the standard case, it is natural to expect that the hedger's price of the call option will depend on the hedger's initial endowment  $x_1$ . To sum up, for each particular market circumstances, the properties of ex-dividend prices may be quite different. Nevertheless, we will argue that most of their properties can be analyzed using general results on BSDEs as a convenient tool.

Recall that  $x_1$  and  $x_2$  stand for the initial endowments of the hedger and the counterparty, respectively. Due to a generic nature of a contract  $(A, C)$ , it is impossible to make any plausible a priori conjectures about relative sizes and/or signs of prices. The equality  $P_t^h(x, A, C) = P_t^c(x, -A, -C)$  means that both parties agree on a common price for the contract. Otherwise, that is, if the equality  $P_t^h(x, A, C) = P_t^c(x, -A, -C)$  fails to hold, then the following situations may arise:

$$(H.1) \quad 0 \leq P_t^c(x_2, -A, -C) < P_t^h(x_1, A, C),$$

$$(H.2) \quad P_t^c(x_1, A, C) \leq 0 < P_t^h(x_2, -A, -C),$$

$$(H.3) \quad P_t^c(x_2, -A, -C) < P_t^h(x_1, A, C) \leq 0,$$

and, symmetrically,

$$(C.1) \quad 0 \leq P_t^h(x_1, A, C) < P_t^c(x_2, -A, -C),$$

$$(C.2) \quad P_t^h(x_1, A, C) \leq 0 < P_t^c(x_2, -A, -C),$$

$$(C.3) \quad P_t^h(x_1, A, C) < P_t^c(x_2, -A, -C) \leq 0.$$

Before analyzing each situation, let us recall that the cash flows of a contract  $(A, C)$  are invariably considered from the perspective of the hedger, so it makes sense to observe that the counterparty faces the cash flows given by  $(-A, -C)$ . Consequently, in case (H.1), we may say that the hedger is the seller of  $(A, C)$  and the counterparty is the buyer of  $(-A, -C)$ , but the counterparty is not willing to pay the amount demanded by the hedger. In case (H.2), both parties are willing to be sellers of the contract, meaning in practice that the hedger is ready to sell  $(A, C)$  and the counterparty is willing to sell  $(-A, -C)$ . Finally, case (H.3) refers to the situation the counterparty is willing to be the seller of  $(-A, -C)$ , whereas the hedger can now be seen as a buyer of  $(A, C)$ , but he is not willing to pay the price that is needed by the counterparty to replicate the contract.

Assume that the market model is arbitrage-free for both parties in the sense of Definition 3.3. Then in all three cases, (H.1)–(H.3), any  $\mathcal{G}_t$ -measurable random variable  $P_t^f$  satisfying

$$P_t^f \in [P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)] \quad (3.9)$$

can be considered to be a *fair price* for both the hedger and his counterparty, in the sense that a bilateral transaction done at the price  $P_t^f$  will not generate an arbitrage opportunity for neither of them. Hence the interval  $[P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)]$  represents the range of fair prices of the contract  $(A, C)$  for both parties, as seen from the perspective of the hedger (a special case of this interval was dubbed the *arbitrage-band* by Bergman [1]).

**Definition 3.10** The  $\mathcal{G}_t$ -measurable interval  $\mathcal{R}_t^f(x_1, x_2) := [P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)]$  is called the *range of fair bilateral prices* at time  $t$  of an OTC contract  $(A, C)$  between the hedger and the counterparty.

Although the analysis for the cases (C.1)–(C.3) can be done analogously, the financial interpretation and conclusions are quite different. In case (C.1), the hedger is willing to be the seller of  $(A, C)$  and the counterparty is willing to be the buyer and he is ready to pay even more than it is

requested by the hedger. In case (C.2), both parties are disposed to be buyers at their respective prices, meaning that each party is ready to pay a positive premium to another. Finally, in case (C.3), the counterparty is willing to be the seller, whereas the hedger can now be seen as a buyer of  $(A, C)$  and he is ready to pay more than it is demanded by the counterparty. Hence for any  $\mathcal{G}_t$ -measurable random variable  $P_t^p$  satisfying

$$P_t^p \in [P_t^h(x_1, A, C), P_t^c(x_2, -A, -C)] \quad (3.10)$$

can be seen as a price at which both parties would be disposed to make the deal with each other. Note that, unless  $P_t^h(x_1, A, C) = P_t^c(x_2, -A, -C)$ , the price  $P_t^p$  is not a fair bilateral price, in the sense explained above, since an arbitrage opportunity arises for at least one party involved when an OTC contract  $(A, C)$  is traded between them at the price  $P_t^p$ . This simple observation motivates the following definition.

**Definition 3.11** Assume that the inequality  $P_t^h(x_1, A, C) \neq P_t^c(x_2, -A, -C)$  holds. Then the  $\mathcal{G}_t$ -measurable interval  $\mathcal{R}_t^p(x_1, x_2) := [P_t^h(x_1, A, C), P_t^c(x_2, -A, -C)]$  is called the *range of bilaterally profitable prices* at time  $t$  of an OTC contract  $(A, C)$  between the hedger and the counterparty.

Note that in our discussion, we dealt in fact with at least three different concepts of arbitrage:

- (A.1) the classic definition of an arbitrage opportunity that may arise by trading in primary assets,
- (A.2) an arbitrage opportunity associated with a long hedged position in some contract combined with a short unhedged position in the same contract; in that case, the contract's price at time 0 is considered to be exogenously given by the market, that is, it is driven by the law of demand and supply (see Definition 3.3 of an arbitrage in regard to a given contract),
- (A.3) an arbitrage opportunity related to the fact that the hedger and the counterparty may require different premia to implement their respective replicating strategies; if this kind of an arbitrage opportunity arises, then it is simultaneously available to both parties involved in an OTC contract with a price negotiated between them (as in Definition 3.11).

Note that in case (C.2) an immediate *reselling arbitrage opportunity* arises for a third party, that is, a trader who could simultaneously ‘purchase’ a contract from one party and ‘resell’ to the other. Specifically, if  $P_t^h(x_1, A, C) \leq 0$  and  $P_t^c(x_2, -A, -C) > 0$ , then a third party can make a deal with the hedger to face  $(-A, -C)$  and receive  $-P_t^h(x_1, A, C) \geq 0$  and, at the same time, enter the contract with the counterparty to face  $(A, C)$  and get  $P_t^c(x_2, -A, -C) > 0$ . This offsetting strategy produces an immediate profit of  $P_t^c(x_2, -A, -C) - P_t^h(x_1, A, C) > 0$  for the third party.

## 4 Pricing BSDEs and Replicating Strategies

Our next aim is to show that the hedger's and counterparty's prices and their replicating strategies can be found by solving suitable BSDEs. For this purpose, we will use some auxiliary results on BSDEs driven by multi-dimensional continuous martingales (see [14] and the references therein). In Propositions 4.1 and 4.2, we will show that if  $x_1 x_2 \geq 0$ , then the prices  $P_t^h(x_1, A, C)$  and  $P_t^c(x_2, -A, -C)$  are given by the solutions of two BSDEs that are driven by either the common  $(\mathbb{P}^l, \mathbb{G})$ -local martingale  $\tilde{S}^{l, \text{cld}}$  (when  $x_1 \geq 0, x_2 \geq 0$ ) or the common  $(\mathbb{P}^b, \mathbb{G})$ -local martingale  $\tilde{S}^{b, \text{cld}}$  (when  $x_1 \leq 0, x_2 \leq 0$ ). By contrast, when the inequality  $x_1 x_2 < 0$  holds, say  $x_1 > 0$  and  $x_2 < 0$ , then the prices are associated with solutions to the two BSDEs driven by  $\tilde{S}^{l, \text{cld}}$  and  $\tilde{S}^{b, \text{cld}}$ , respectively. Therefore, to find the range of fair (or profitable) bilateral prices using the comparison theorem for BSDEs, we will first need to find a suitable variant of the pricing BSDE for both parties, which will be driven by a common continuous local martingale (see Section 5.2).

### 4.1 Modeling of Risky Assets

To show the existence of a solution to the pricing BSDE, we need to complement Assumptions 3.1 and 3.2 by imposing specific conditions on the underlying market model. For any  $d \times d$  matrix  $m$ ,

the norm of  $m$  is given by  $\|m\|^2 := \text{Tr}(mm^*)$ . In the next assumption, the superscript  $k$  stands for either  $l$  or  $b$ .

**Assumption 4.1** We postulate that:

- (i) the process  $\tilde{S}^{\text{cld}}$  is a continuous, square-integrable,  $(\tilde{\mathbb{P}}^k, \mathbb{G})$ -martingale and has the predictable representation property (PRP) with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^k$ ,
- (ii) there exists an  $\mathbb{R}^{d \times d}$ -valued,  $\mathbb{G}$ -adapted process  $m^k$  such that

$$\langle \tilde{S}^{k, \text{cld}} \rangle_t = \int_0^t m_u^k (m_u^k)^* du \quad (4.1)$$

where  $m^k (m^k)^*$  is invertible and there exists a constant  $K_m > 0$  such that, for all  $t \in [0, T]$ ,

$$\|m_t^k\| + \|(m_t^k (m_t^k)^*)^{-\frac{1}{2}}\| \leq K_m, \quad (4.2)$$

- (iii) the price processes  $S^i$ ,  $i = 1, 2, \dots, d$  of risky assets are bounded.

Note that condition (4.2) means that the process  $m^l$  satisfies Assumption 5.2 in [14]. Although the postulate that the prices  $S^i$ ,  $i = 1, 2, \dots, d$  are bounded can be seen as a quite reasonable real-world requirement, it is rarely satisfied in commonly used financial models, including the classic Black and Scholes model. It is also worth noting that condition (4.2) could appear to be too restrictive. In order to relax this condition, we will need to impose stronger conditions on the process  $\langle \tilde{S}^{l, \text{cld}} \rangle$ . Specifically, we define the matrix-valued process  $\mathbb{S}$

$$\mathbb{S}_t := \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & S_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_t^d \end{pmatrix}.$$

**Definition 4.1** We say that  $\gamma$  satisfies the *ellipticity* condition if there exists a constant  $\Lambda > 0$

$$\sum_{i,j=1}^d (\gamma_t \gamma_t^*)_{ij} a_i a_j \geq \Lambda |a|^2 = \Lambda a^* a, \quad \text{for all } a \in \mathbb{R}^d \text{ and } t \in [0, T]. \quad (4.3)$$

We consider the following assumption, which should be seen as an alternative to Assumption 4.1. Once again, the superscript  $k$  is equal either to  $l$  or  $b$ .

**Assumption 4.2** We postulate that:

- (i) the process  $\tilde{S}^{k, \text{cld}}$  is a continuous, square-integrable,  $(\tilde{\mathbb{P}}^k, \mathbb{G})$ -martingale and has the PRP with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^k$ ,
- (ii) equality (4.1) holds with the  $\mathbb{G}$ -adapted process  $m^k$  such that  $m^k (m^k)^*$  is invertible and satisfies  $m^k (m^k)^* = \mathbb{S} \gamma \gamma^* \mathbb{S}$  where a  $d$ -dimensional square matrix  $\gamma$  of  $\mathbb{G}$ -adapted processes satisfies the ellipticity condition (4.3).

**Remark 4.1** We will show that Assumption 4.1 or 4.2 can be easily met when the prices of risky assets are given by the diffusion-type model. For example, we may assume that each risky asset  $S^i$ ,  $i = 1, 2, \dots, d$  has the ex-dividend price dynamics under  $\mathbb{P}$  given by

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right), \quad S_0^i > 0,$$

or, equivalently, the  $d$ -dimensional process  $S = (S^1, \dots, S^d)^*$  satisfies

$$dS_t = \mathbb{S}_t (\mu_t dt + \sigma_t dW_t)$$

where  $W = (W^1, \dots, W^d)^*$  is the  $d$ -dimensional Brownian motion,  $\mu = (\mu^1, \dots, \mu^d)^*$  is an  $\mathbb{R}^d$ -valued,  $\mathbb{F}^W$ -adapted process,  $\sigma = [\sigma^{ij}]$  is a  $d$ -dimensional square matrix of  $\mathbb{F}^W$ -adapted processes satisfying the *ellipticity* condition. We now set  $\mathbb{G} = \mathbb{F}^W$  and we recall that the  $d$ -dimensional Brownian motion  $W$  enjoys the predictable representation property with respect to its natural filtration  $\mathbb{F}^W$ ; this property is shared by the process  $\widetilde{W}$  defined (4.5).

Assuming that the corresponding dividend processes are given by  $A_t^i = \int_0^t \kappa_u^i S_u^i du$ , we obtain

$$d\widetilde{S}_t^{i,l,\text{cld}} = (B_t^l)^{-1} (dS_t^i + dA_t^i - r_t^l S_t^i dt) = (B_t^l)^{-1} S_t^i \left( (\mu_t^i + \kappa_t^i - r_t^l) dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right).$$

If we denote  $S^{l,\text{cld}} = (S^{1,l,\text{cld}}, \dots, S^{d,l,\text{cld}})^*$  and  $\mu + \kappa - r^l = (\mu^1 + \kappa^1 - r^l, \dots, \mu^d + \kappa^d - r^l)^*$ , then

$$d\widetilde{S}_t^{l,\text{cld}} = (B_t^l)^{-1} \mathbb{S}_t \left( (\mu_t + \kappa_t - r_t^l) dt + \sigma_t dW_t \right).$$

We set  $a_t := \sigma_t^{-1}(\mu_t + \kappa_t - r_t^l)$  for all  $t \in [0, T]$  and we define the probability measure  $\widetilde{\mathbb{P}}^l$  on  $(\Omega, \mathcal{F}_T^W)$  by

$$\frac{d\widetilde{\mathbb{P}}^l}{d\mathbb{P}} = \exp \left\{ - \int_0^T a_t dW_t - \frac{1}{2} \int_0^T |a_t|^2 dt \right\}. \quad (4.4)$$

Then  $\widetilde{\mathbb{P}}^l$  is equivalent to  $\mathbb{P}$  and, from the Girsanov theorem, the process  $\widetilde{W} := (\widetilde{W}^1, \widetilde{W}^2, \dots, \widetilde{W}^d)^*$  is a Brownian motion under  $\widetilde{\mathbb{P}}^l$ , where

$$d\widetilde{W}_t := dW_t + a_t dt = dW_t + \sigma_t^{-1}(\mu_t + \kappa_t - r_t^l) dt. \quad (4.5)$$

It is clear that under  $\widetilde{\mathbb{P}}^l$

$$d\widetilde{S}_t^{l,\text{cld}} = (B_t^l)^{-1} \mathbb{S}_t \sigma_t d\widetilde{W}_t.$$

Therefore, if the processes  $\mu$ ,  $\sigma$  and  $\kappa$  are bounded, then the processes  $\widetilde{S}^{i,l,\text{cld}}$ ,  $i = 1, 2, \dots, d$  are continuous, square-integrable,  $(\widetilde{\mathbb{P}}^l, \mathbb{G})$ -martingales. Furthermore, the quadratic variation of  $\widetilde{S}^{l,\text{cld}}$  equals

$$\langle \widetilde{S}^{l,\text{cld}} \rangle_t = \int_0^t m_u^l (m_u^l)^* du$$

where  $m^l (m^l)^* = \mathbb{S} \gamma \gamma^* \mathbb{S}$  and  $\gamma := (B^l)^{-1} \sigma$ . Obviously,  $m^l (m^l)^*$  is invertible and thus Assumption 4.2 is satisfied. Moreover, if the processes  $S^i$ ,  $i = 1, 2, \dots, d$  are bounded, then the process  $m^l$  satisfies condition (4.2) and thus Assumption 4.1 is valid with  $k = l$ .

## 4.2 Hedgers's Prices and Replicating Strategies

From now on, we work under the standing assumption that  $Q_t = t$  for every  $t \in [0, T]$  in Assumption 3.1 in [14] and thus also in all results in Sections 3–5 of [14] (in particular, in the definition of the norm for the space  $\widehat{\mathcal{H}}_{\lambda}^{2,d}$ ). Note that this postulate is consistent with either of Assumptions 4.1 and 4.2. Moreover, we henceforth postulate that the processes  $r^l$ ,  $r^b$  and  $r^{i,b}$  for  $i = 1, 2, \dots, d$  are nonnegative and bounded.

The following result describes the prices and hedging strategies for the hedger. Recall that  $A^C := A + C + F^C$  and

$$F_t^C = - \int_0^t (B_u^{c,b})^{-1} C_u^+ dB_u^{c,b} + \int_0^t (B_u^{c,l})^{-1} C_u^- dB_u^{c,l}.$$

Following [14], but with  $Q_t = t$ , we denote by  $\widehat{\mathcal{H}}_0^{2,d}$  the subspace of all  $\mathbb{R}^d$ -valued,  $\mathbb{G}$ -adapted processes  $X$  with

$$|X|_{\widehat{\mathcal{H}}_0^{2,d}}^2 := \mathbb{E}_{\mathbb{P}} \left[ \int_0^T \|X_t\|^2 dt \right] < \infty. \quad (4.6)$$

Also, let  $\widehat{L}_0^2$  stand for the space of all real-valued,  $\mathcal{G}_T$ -measurable random variables  $\eta$  such that  $|\eta|_{\widehat{L}_0^2}^2 = \mathbb{E}_{\mathbb{P}}(\eta^2) < \infty$ .

**Definition 4.2** A contract  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^l$*  if the process  $A^{C,l}$  belongs to  $\widehat{\mathcal{H}}_0^2$  and the random variable  $A_T^{C,l}$  belongs to  $\widehat{L}_0^2$  under  $\tilde{\mathbb{P}}^l$ . A contract  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^b$*  if the process  $A^{C,b}$  belongs to  $\widehat{\mathcal{H}}_0^2$  and the random variable  $A_T^{C,b}$  belongs to  $\widehat{L}_0^2$  under  $\tilde{\mathbb{P}}^b$ .

**Proposition 4.1** (i) Let either Assumption 4.1 or Assumption 4.2 with  $k = l$  be satisfied. Then for any real number  $x \geq 0$  and any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^l$ , the hedger's ex-dividend price satisfies  $P^h(x, A, C) = B^l(Y^{h,l,x} - x) - C$  where  $(Y^{h,l,x}, Z^{h,l,x})$  is the unique solution to the BSDE

$$\begin{cases} dY_t^{h,l,x} = Z_t^{h,l,x,*} d\tilde{S}_t^{l,cl} + \tilde{f}_l(t, Y_t^{h,l,x}, Z_t^{h,l,x}) dt + dA_t^{C,l}, \\ Y_T^{h,l,x} = x. \end{cases} \quad (4.7)$$

The unique replicating strategy equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where, for every  $t \in [0, T]$  and  $i = 1, 2, \dots, d$ ,

$$\xi_t^i = Z_t^{h,l,x,i}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+, \quad \eta_t^b = -(B_t^{c,b})^{-1}C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1}C_t^-,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( B_t^l Y_t^{h,l,x} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( B_t^b Y_t^{h,l,x} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-. \end{aligned}$$

(ii) Let either Assumption 4.1 or Assumption 4.2 with  $k = b$  be satisfied. Then for any real number  $x \leq 0$  and any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^b$ , the hedger's ex-dividend price satisfies  $P^h(x, A, C) = B^b(Y^{h,b,x} - x) - C$  where  $(Y^{h,b,x}, Z^{h,b,x})$  is the unique solution to the BSDE

$$\begin{cases} dY_t^{h,b,x} = Z_t^{h,b,x,*} d\tilde{S}_t^{b,cl} + \tilde{f}_b(t, Y_t^{h,b,x}, Z_t^{h,b,x}) dt + dA_t^{C,b}, \\ Y_T^{h,b,x} = x. \end{cases} \quad (4.8)$$

The unique replicating strategy equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where, for every  $t \in [0, T]$  and  $i = 1, 2, \dots, d$ ,

$$\xi_t^i = Z_t^{h,b,x,i}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+, \quad \eta_t^b = -(B_t^{c,b})^{-1}C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1}C_t^-,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( B_t^l Y_t^{h,b,x} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( B_t^b Y_t^{h,b,x} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-. \end{aligned}$$

*Proof.* Assume first that  $x \geq 0$ . Then, from Theorems 4.1 and 5.1 in [14], we know that if either Assumption 4.1 or Assumption 4.2 with  $k = l$  is satisfied and  $A^{C,l} \in \widehat{\mathcal{H}}_0^2$  and  $A_T^{C,l} \in \widehat{L}_0^2$  under  $\tilde{\mathbb{P}}^l$ , then BSDE (4.7) has a unique solution  $(Y^{h,l,x}, Z^{h,l,x})$ . Thus, from Proposition 5.2 in [2] we obtain  $P^h(x, A, C) = B^l(Y^{h,l,x} - x) - C$ . Moreover, the replicating strategy  $\varphi$  can be constructed uniquely, as was explained in Section 2.3. In view of Remark 5.3 in [2], an analogous analysis can be done when the initial endowment satisfies  $x \leq 0$ .  $\square$

**Remark 4.2** Let us give some comments on the uniqueness of a replicating strategy in Proposition 4.1. We only consider the case when  $x \geq 0$ , since similar arguments apply to the case  $x \leq 0$ . The uniqueness of the solution of BSDE (4.7) means that if  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  are two solutions of BSDE (4.7), then

$$\mathbb{E}_{\tilde{\mathbb{P}}^l} \left[ \int_0^T |Y_t^1 - Y_t^2|^2 dt + \int_0^T \|(m_t^l)^* Z_t^1 - (m_t^l)^* Z_t^2\|^2 dt \right] = 0. \quad (4.9)$$

Under Assumption 4.1, there exists a constant  $k_m$  such that  $\|(m_t^l)^*\| \geq k_m$  and a constant  $K$  such that  $\|\mathbb{S}_t\| \leq K$ . Therefore, from (4.9) we deduce that

$$\mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T \|\mathbb{S}_t Z_t^1 - \mathbb{S}_t Z_t^2\|^2 dt \right] = 0. \quad (4.10)$$

Under Assumption 4.2 with  $k = l$ , we have that  $m^l(m^l)^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$  and thus

$$\mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T \|(m_t^l)^*(Z_t^1 - Z_t^2)\|^2 dt \right] = \mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T (Z_t^1 - Z_t^2)^* \mathbb{S}\gamma\gamma^*\mathbb{S}(Z_t^1 - Z_t^2) dt \right].$$

Since  $\gamma$  satisfies the ellipticity condition, there exists a constant  $\Lambda > 0$  such that

$$\mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T (Z_t^1 - Z_t^2)^* \mathbb{S}_t \gamma \gamma^* \mathbb{S}_t (Z_t^1 - Z_t^2) dt \right] \geq \Lambda \mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T \|\mathbb{S}_t Z_t^1 - \mathbb{S}_t Z_t^2\|^2 dt \right].$$

We conclude that under either of Assumptions 4.1 and 4.2 with  $k = l$ , equality (4.10) is satisfied by any two solutions of BSDE (4.7).

From the above arguments and the structure of the replicating strategy (see Proposition 4.1)

$$\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l),$$

we know that the uniqueness is in the sense of equivalence with respect to  $\mathbb{P}_l \otimes \ell$ . Moreover, for  $\xi = (\xi^1, \dots, \xi^d)^*$ ,  $\psi^l$ ,  $\psi^b$  and  $\psi = (\psi^{1,b}, \dots, \psi^{d,b})^*$  the uniqueness holds in the following norm

$$\|\varphi\| := \mathbb{E}_{\tilde{\mathbb{P}}_l} \left[ \int_0^T \|\mathbb{S}_t \xi_t\|^2 dt + \int_0^T (|\psi_t^l|^2 + |\psi_t^b|^2) dt + \int_0^T \|\psi_t\|^2 dt \right].$$

### 4.3 Counterparty's Prices and Replicating Strategies

Let us first observe that, in view of Assumption 2.5, we have  $(-A)^{-C} = -A^C$ . Using Definition 3.8, one can prove the following result for the counterparty.

**Proposition 4.2** *Let the assumptions of part (i) or (ii) in Proposition 4.1 be satisfied for  $x \geq 0$  and  $x \leq 0$ , respectively. Then the counterparty's ex-dividend price satisfies, for every  $t \in [0, T]$ ,*

$$P_t^c(x, -A, -C) = - \left( B_t^l(Y_t^{c,l,x} - x) + C_t \right) \mathbf{1}_{\{x \geq 0\}} - \left( B_t^b(Y_t^{c,b,x} - x) + C_t \right) \mathbf{1}_{\{x \leq 0\}}$$

where  $(Y_t^{c,l,x}, Z_t^{c,l,x})$  and  $(Y_t^{c,b,x}, Z_t^{c,b,x})$  is respectively the unique solution to the BSDE

$$\begin{cases} dY_t^{c,l,x} = Z_t^{c,l,x,*} d\tilde{S}_t^{l,cl,d} + \tilde{f}_l(t, Y_t^{c,l,x}, Z_t^{c,l,x}) dt - dA_t^{C,l}, \\ Y_T^{c,l,x} = x, \end{cases} \quad (4.11)$$

and

$$\begin{cases} dY_t^{c,b,x} = Z_t^{c,b,x,*} d\tilde{S}_t^{b,cl,d} + \tilde{f}_b(t, Y_t^{c,b,x}, Z_t^{c,b,x}) dt - dA_t^{C,b}, \\ Y_T^{c,b,x} = x. \end{cases} \quad (4.12)$$

The unique replicating strategy equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where, for every  $t \in [0, T]$  and  $i = 1, 2, \dots, d$ ,

$$\xi_t = Z_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + Z_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+, \quad \eta_t^b = -(B_t^{c,b})^{-1} C_t^-, \quad \eta_t^l = (B_t^{c,l})^{-1} C_t^+,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( B_t^l Y_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + B_t^b Y_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( B_t^l Y_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + B_t^b Y_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-. \end{aligned}$$

## 5 Properties of Arbitrage Prices

Recall that we consider the special case of an exogenous margin account with rehypothesized cash collateral. The exogenous property implies that  $C$  does not depend on a strategy  $\varphi$  and the value of the strategy. We denote the initial endowment of the hedger (resp., counterparty) by  $x_1$  (resp.,  $x_2$ ). We will examine the pricing and hedging problems for both parties in the following situations:

- the initial endowments satisfy  $x_1 \geq 0$  and  $x_2 \geq 0$ ,
- the initial endowments satisfy  $x_1 \leq 0$  and  $x_2 \leq 0$ ,
- the initial endowments satisfy  $x_1 x_2 \leq 0$ .

Our goal is to establish inequalities for unilateral prices for each of the three above-mentioned cases (see Propositions 5.1, 5.2 and 5.4, respectively) and thus also to derive the ranges of fair bilateral prices. In the last case, that is, when  $x_1 x_2 \leq 0$  we also examine the properties of the class of *monotone* contracts (see Section 5.2.2). Finally, we study the monotonicity of unilateral prices with respect to the initial endowment and we derive the pricing PDE in the Markovian framework.

### 5.1 Initial Endowments of Equal Signs

We first assume that both parties have positive initial endowments, that is,  $x_1 \geq 0$  and  $x_2 \geq 0$ . The proof of Proposition 5.1 is provided in Nie and Rutkowski [14] (see Theorem 5.2 in [14]), where a suitable comparison theorem for BSDEs driven by a multi-dimensional martingale is also proven.

**Proposition 5.1** *Let either Assumption 4.1 or Assumption 4.2 with  $k = l$  hold. If  $x_1 \geq 0$  and  $x_2 \geq 0$ , then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^l$  we have, for all  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.}, \quad (5.1)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

In the second step, we postulate that both parties have positive initial endowments, that is,  $x_1 \leq 0$  and  $x_2 \leq 0$ . As was explained in Remark 3.1, we now need assume that  $r_t^b \leq r_t^{i,b}$  for  $i = 1, 2, \dots, d$ . The proof of Proposition 5.2 is postponed to the appendix.

**Proposition 5.2** *Let either Assumption 4.1 or Assumption 4.2 with  $k = b$  hold. If  $x_1 \leq 0$ ,  $x_2 \leq 0$  and  $r^b \leq r^{i,b}$  for  $i = 1, 2, \dots, d$ , then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^b$  we have, for all  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^b - \text{a.s.}, \quad (5.2)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

### 5.2 Initial Endowments of Opposite Signs

We now consider the case when the initial endowments of the two parties have opposite signs, specifically, we postulate that  $x_1 \geq 0$  and  $x_2 \leq 0$ . From Propositions 4.1 and 4.2, it follows that  $P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$  where  $(Y^{h,l,x_1}, Z^{h,l,x_1})$  is the unique solution of the BSDE

$$\begin{cases} dY_t^{h,l,x_1} = Z_t^{h,l,x_1,*} d\tilde{S}_t^{l,\text{cld}} + \tilde{f}_l(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt + dA_t^{C,l}, \\ Y_T^{h,l,x_1} = x_1, \end{cases}$$

and  $P^c(x_2, -A, -C) = -(B^b(Y^{c,b,x_2} - x_2) + C)$  where  $(Y^{c,b,x_2}, Z^{c,b,x_2})$  is the unique solution of the BSDE

$$\begin{cases} dY_t^{c,b,x_2} = Z_t^{c,b,x_2,*} d\tilde{S}_t^{b,\text{cld}} + \tilde{f}_b(t, Y_t^{c,b,x_2}, Z_t^{c,b,x_2}) dt - dA_t^{C,b}, \\ Y_T^{c,b,x_2} = x_2. \end{cases}$$



Note that the BSDE for  $Y^{h,l,x_1}$  is driven by the  $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -local martingale  $\tilde{S}^{l,\text{cld}}$ , but the BSDE for  $Y^{c,b,x_2}$  is driven by the  $(\tilde{\mathbb{P}}^b, \mathbb{G})$ -local martingale  $\tilde{S}^{b,\text{cld}}$ . We will now attempt to find another probability measure  $\tilde{\mathbb{P}}$  equivalent to  $\mathbb{P}$  and a  $(\tilde{\mathbb{P}}, \mathbb{G})$ -local martingale  $\tilde{S}^{\text{cld}}$  such that the BSDEs related to  $P^h(x_1, A, C)$  and  $P^c(x_2, A, C)$  are both driven by a common  $(\tilde{\mathbb{P}}^b, \mathbb{G})$ -local martingale  $\tilde{S}^{\text{cld}}$ . If we denote  $\tilde{Y}^{h,l,x_1} = B^l(Y^{h,l,x_1} - x_1)$ , then  $P^h(x_1, A, C) = \tilde{Y}^{h,l,x_1} - C$ . In view of (2.14) and (2.16), we obtain

$$\begin{aligned}
d\tilde{Y}_t^{h,l,x_1} &= -x_1 dB_t^l + Y_t^{h,l,x_1} dB_t^l + B_t^l dY_t^{h,l,x_1} \\
&= -x_1 r_t^l B_t^l dt + r_t^l B_t^l Y_t^{h,l,x_1} dt + B_t^l Z_t^{h,l,x_1,*} d\tilde{S}_t^{l,\text{cld}} + B_t^l f_l(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt + dA_t^C \\
&= -x_1 r_t^l B_t^l dt + r_t^l B_t^l Y_t^{h,l,x_1} dt + \sum_{i=1}^d Z_t^{h,l,x_1,i} (dS_t^i - r_t^l S_t^i dt + dA_t^i) + f_l(t, B_t^l Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt \\
&\quad - r_t^l B_t^l Y_t^{h,l,x_1} dt + dA_t^C \\
&= -x_1 r_t^l B_t^l dt + \sum_{i=1}^d Z_t^{h,l,x_1,i} (dS_t^i - r_t^l S_t^i dt + dA_t^i) \\
&\quad + \sum_{i=1}^d r_t^l Z_t^{h,l,x_1,i} S_t^i dt - \sum_{i=1}^d r_t^{i,b} (Z_t^{h,l,x_1,i} S_t^i)^+ dt + r_t^l \left( B_t^l Y_t^{h,l,x_1} + \sum_{i=1}^d (Z_t^{h,l,x_1,i} S_t^i)^- \right)^+ dt \\
&\quad - r_t^l \left( B_t^l Y_t^{h,l,x_1} + \sum_{i=1}^d (Z_t^{h,l,x_1,i} S_t^i)^- \right)^- dt + dA_t^C \\
&= -x_1 r_t^l B_t^l dt + \sum_{i=1}^d Z_t^{h,l,x_1,i} (dS_t^i + dA_t^i) + g(t, B_t^l Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt + dA_t^C
\end{aligned}$$

where

$$g(t, y, z) = -\sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^-. \quad (5.3)$$

Upon denoting  $\tilde{Z}^{h,l,x_1} = Z^{h,l,x_1}$ , we obtain

$$d\tilde{Y}_t^{h,l,x_1} = \sum_{i=1}^d \tilde{Z}_t^{h,l,x_1,i} (dS_t^i + dA_t^i) - x_1 r_t^l B_t^l dt + g(t, \tilde{Y}_t^{h,l,x_1} + x_1 B_t^l, \tilde{Z}_t^{h,l,x_1}) dt + dA_t^C.$$

Similarly, if we denote  $\tilde{Y}^{c,b,x_2} = -B^b(Y^{c,b,x_2} - x_2)$  and  $\tilde{Z}^{c,b,x_2} = -Z^{c,b,x_2}$ , then the counterparty's price equals  $P^c(x_2, A, C) = \tilde{Y}^{c,b,x_2} - C$ . In view of (2.15), (2.17) and (5.3), we obtain

$$\begin{aligned}
d\tilde{Y}_t^{c,b,x_2} &= x_2 dB_t^b - Y_t^{c,b,x_2} dB_t^b - B_t^b dY_t^{c,b,x_2} \\
&= x_2 r_t^b B_t^b dt - r_t^b B_t^b Y_t^{c,b,x_2} dt - B_t^b Z_t^{c,b,x_2,*} d\tilde{S}_t^{b,\text{cld}} - B_t^b f_b(t, Y_t^{c,b,x_2}, Z_t^{c,b,x_2}) dt + dA_t^C \\
&= x_2 r_t^b B_t^b dt - r_t^b B_t^b Y_t^{c,b,x_2} dt - \sum_{i=1}^d Z_t^{c,b,x_2,i} (dS_t^i - r_t^b S_t^i dt + dA_t^i) - f_b(t, B_t^b Y_t^{c,b,x_2}, Z_t^{c,b,x_2}) dt \\
&\quad + r_t^b B_t^b Y_t^{c,b,x_2} dt + dA_t^C \\
&= x_2 r_t^b B_t^b dt - \sum_{i=1}^d Z_t^{c,b,x_2,i} (dS_t^i + dA_t^i) - g(t, B_t^b Y_t^{c,b,x_2}, Z_t^{c,b,x_2}) dt + dA_t^C \\
&= \sum_{i=1}^d \tilde{Z}_t^{c,b,x_2,i} (dS_t^i + dA_t^i) + x_2 r_t^b B_t^b dt - g(t, -\tilde{Y}_t^{c,b,x_2} + x_2 B_t^b, -\tilde{Z}_t^{c,b,x_2}) dt + dA_t^C.
\end{aligned}$$

The following assumptions are motivated by Assumptions 4.1 and 4.2, respectively.

**Assumption 5.1** We postulate that:

(i) there exists a probability measure  $\tilde{\mathbb{P}}^\beta$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,\text{cld}}$ ,  $i = 1, 2, \dots, d$  given by

$$d\tilde{S}_t^{i,\text{cld}} = dS_t^i + dA_t^i - \beta_t^i S_t^i dt \quad (5.4)$$

for some  $\mathbb{G}$ -adapted processes  $\beta^i$  satisfying  $r^b \leq \beta^i \leq r^{i,b}$ , are continuous, square-integrable  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -martingales, and have the PRP with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^\beta$ ,

(ii) there exists an  $\mathbb{R}^{d \times d}$ -valued,  $\mathbb{G}$ -adapted process  $m$  such that

$$\langle \tilde{S}^{\text{cld}} \rangle_t = \int_0^t m_u m_u^* du, \quad (5.5)$$

where  $m(m)^*$  is invertible and there exists a constant  $K_m > 0$  such that, for all  $t \in [0, T]$ ,

$$\|m_t\| + \|(m_t(m_t)^*)^{-\frac{1}{2}}\| \leq K_m, \quad (5.6)$$

(iii) the price processes  $S^i$ ,  $i = 1, 2, \dots, d$  of risky assets are bounded.

**Assumption 5.2** We postulate that:

- (i) there exists a probability measure  $\tilde{\mathbb{P}}^\beta$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,\text{cld}}$ ,  $i = 1, 2, \dots, d$  given by (5.4) are  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -continuous square integrable martingales, and have the PRP with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^\beta$ ,
- (ii) condition (5.5) holds with the  $\mathbb{G}$ -adapted process  $m$  such that  $mm^*$  is invertible and given by  $mm^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$  where a  $d$ -dimensional square matrix  $\gamma$  of  $\mathbb{G}$ -adapted processes satisfies the ellipticity condition (4.3).

**Remark 5.1** Recall that

$$d\tilde{S}_t^{i,l,\text{cld}} = (B_t^l)^{-1} (dS_t^i - r_t^l S_t^i dt + dA_t^i), \quad d\tilde{S}_t^{i,b,\text{cld}} = (B_t^b)^{-1} (dS_t^i - r_t^b S_t^i dt + dA_t^i).$$

Since  $r^l$  is non-negative and bounded, we obtain  $C_0 < (B^l)^{-1} < 1$  for some constant  $C_0$ . Then Assumption 5.1 with  $\beta^i = r^l$  is equivalent to Assumption 3.1. Similar comments apply to other assumptions. We mention that, from Proposition 3.2 and  $r^b \leq \beta^i \leq r^{i,b}$ , we know that under Assumption 5.1 or Assumption 5.2, our partial netting model is arbitrage-free with respect to any contract for both the hedger and counterparty with  $x_1, x_2 \in \mathbb{R}$ .

**Remark 5.2** The above assumptions can be easily satisfied for the diffusion-type market model similarly to the one in Remark 4.1; the details are left to the reader.

**Definition 5.1** We say that  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^\beta$*  when  $A^C \in \hat{\mathcal{H}}_0^2$  and  $A_T^C \in \hat{L}_0^2$  under  $\tilde{\mathbb{P}}^\beta$ .

For  $g$  given by (5.3), let us define

$$g^h(t, x, y, z) := \sum_{i=1}^d z_t^i \beta_t^i S_t^i + (-xr_t^l B_t^l + g(t, y + xB_t^l, z))\mathbf{1}_{\{x \geq 0\}} + (-xr_t^b B_t^b + g(t, y + xB_t^b, z))\mathbf{1}_{\{x \leq 0\}}$$

and

$$g^c(t, x, y, z) := \sum_{i=1}^d z_t^i \beta_t^i S_t^i + (xr_t^l B_t^l - g(t, -y + xB_t^l, -z))\mathbf{1}_{\{x \geq 0\}} + (xr_t^b B_t^b - g(t, -y + xB_t^b, -z))\mathbf{1}_{\{x \leq 0\}}.$$

The next result is a counterpart of Propositions 4.1 and 4.2. In view of the discussion at the beginning of this subsection, Proposition 5.3 is a rather straightforward consequence of Theorem 4.1 in [14] and thus its proof is omitted.

**Proposition 5.3** *Let either Assumption 5.1 or Assumption 5.2 be valid. Consider an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ . Then  $P^h(x_1, A, C) = \tilde{Y}^{h,x_1} - C$  and  $P^c(x_2, -A, -C) = \tilde{Y}^{c,x_2} - C$  where  $(\tilde{Y}^{h,x_1}, \tilde{Z}^{h,x_1})$  is the unique solution of the BSDE*

$$\begin{cases} d\tilde{Y}_t^{h,x_1} = \tilde{Z}_t^{h,x_1,*} d\tilde{S}_t^{\text{cld}} + g^h(t, x_1, \tilde{Y}_t^{h,x_1}, \tilde{Z}_t^{h,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,x_1} = 0, \end{cases} \quad (5.7)$$

and  $(\tilde{Y}^{c,x_2}, \tilde{Z}^{c,x_2})$  is the unique solution of the BSDE

$$\begin{cases} d\tilde{Y}_t^{c,x_2} = \tilde{Z}_t^{c,x_2,*} d\tilde{S}_t^{\text{cld}} + g^c(t, x_2, \tilde{Y}_t^{c,x_2}, \tilde{Z}_t^{c,x_2}) dt + dA_t^C, \\ \tilde{Y}_T^{c,x_2} = 0, \end{cases} \quad (5.8)$$

Moreover, the unique replicating strategy for the hedger equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where for every  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = \tilde{Z}_t^{h, x_1, i}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+, \quad \eta_t^b = -(B_t^{c,b})^{-1}C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1}C_t^-,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( \tilde{Y}_t^{h, x_1} + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( \tilde{Y}_t^{h, x_1} + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-. \end{aligned}$$

The unique replicating strategy for the counterparty equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where for every  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = -\tilde{Z}_t^{c, x_2, i}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+, \quad \eta_t^b = -(B_t^{c,b})^{-1}C_t^-, \quad \eta_t^l = (B_t^{c,l})^{-1}C_t^+.$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( -\tilde{Y}_t^{c, x_2} + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( -\tilde{Y}_t^{c, x_2} + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} + \sum_{i=1}^d (\xi_t^i S_t^i)^- \right)^-. \end{aligned}$$

One can check that  $g^h(t, x, 0, 0) = g^c(t, x, 0, 0) = 0$  for all  $x \in \mathbb{R}$ . Consider any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ . If, in addition,  $A^C$  is a decreasing process, then for any  $x_1, x_2 \in \mathbb{R}$ ,  $\tilde{Y}^{h, x_1} \geq 0$  and  $\tilde{Y}^{c, x_2} \geq 0$ , where  $(\tilde{Y}^{h, x_1}, \tilde{Z}^{h, x_1})$  is the unique solution of BSDE (5.7) and  $(\tilde{Y}^{c, x_2}, \tilde{Z}^{c, x_2})$  is the unique solution of BSDE (5.8). Consequently,  $P^h(x_1, A, C) \geq -C$  and  $P^c(x_2, -A, -C) \geq -C$ . If the process  $A^C$  is increasing, then for any  $x_1, x_2 \in \mathbb{R}$  we have  $\tilde{Y}^{h, x_1} \leq 0$  and  $\tilde{Y}^{c, x_2} \leq 0$ , so that  $P^h(x_1, A, C) \leq -C$  and  $P^c(x_2, -A, -C) \leq -C$ .

**Example 5.1** In Example 3.1, we considered a contract  $(A, C)$  with  $A_t = p \mathbf{1}_{[0, T]}(t) + X \mathbf{1}_{[T]}(t)$  and  $C = 0$ . Let us first assume that  $X \leq 0$ ; for instance, for a European call option  $X = -(S_T^i - K)^+$  and for a European put option  $X = -(K - S_T^i)^+$ . Then, obviously, the process  $A^C - A_0 = A - A_0$  is decreasing. Then, for any  $x \in \mathbb{R}$ , both  $P_t^h(x, A, C)$  and  $P_t^c(x, -A, -C)$  are positive, meaning that the hedger is the seller and the counterparty is the buyer. Similarly, if  $X \geq 0$ , for instance, for a European call option  $X = (S_T^i - K)^+$  and for European put option  $X = (K - S_T^i)^+$ , then, for any  $x \in \mathbb{R}$ , both  $P_t^h(x, A, C)$  and  $P_t^c(x, -A, -C)$  are negative, meaning that the counterparty is the seller and the hedger is the buyer. Needless to say that such properties of unilateral options prices were expected.

### 5.2.1 General Contracts

Since  $\tilde{S}^{i, \text{cld}}, i = 1, 2, \dots, d$  are  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -local martingales under Assumption 5.1, we can apply the comparison theorem to BSDEs (5.7) and (5.8) in order to establish the following proposition (for the proof, see Section 6).

**Proposition 5.4** *Let either Assumption 5.1 or Assumption 5.2 be valid. Assume that  $x_1 \geq 0, x_2 \leq 0$  and  $r^b \leq r^{i,b}$  for  $i = 1, 2, \dots, d$ . Then the following statement are valid.*

(i) *If  $x_1 x_2 = 0$ , then for an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}, \quad (5.9)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

(ii) *Let  $r^l$  and  $r^b$  be deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . Then inequality (5.9) holds for all contracts  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$  if and only if  $x_1 x_2 = 0$ .*

Notice that if  $x_1 x_2 = 0$  then, from Propositions 5.1 and 5.2, we know that the desired inequality holds under respective assumptions. However, in the current proposition, we are working under Assumption 5.1, so that it is not clear whether the pricing inequality still holds.

Finally, for the case  $x_1 \leq 0, x_2 \geq 0$ , one can show that the following result is valid. The proof of Proposition 5.5 is similar to that of Proposition 5.4 and thus it is omitted.

**Proposition 5.5** *Let either Assumption 5.1 or Assumption 5.2 be valid. Assume that  $x_1 \leq 0, x_2 \geq 0$  and  $r^b \leq r^{i,b}$  for  $i = 1, 2, \dots, d$ . Then the following statements are valid.*  
*(i) If  $x_1 x_2 = 0$ , then for an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}, \quad (5.10)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

*(ii) Let  $r^l$  and  $r^b$  be deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . Then inequality (5.10) holds for all contracts  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$  if and only if  $x_1 x_2 = 0$ .*

**Remark 5.3** Under the assumptions of Proposition 5.4, one can prove that Propositions 5.1 and 5.2 hold under  $\tilde{\mathbb{P}}^\beta$ , that is, if  $x_1 x_2 \geq 0$ , then for any  $t \in [0, T]$ ,

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.} \quad (5.11)$$

Indeed, using similar arguments as in the proof of Proposition 5.4, one can show that

$$g^h(t, x_1, y, z) - g^c(t, x_2, y, z) \leq \sum_{i=1}^d |z^i S_t^i| ((r_t^l - r_t^{i,b}) \mathbf{1}_{\{x_1 \geq 0, x_2 \geq 0\}} + (r_t^b - r_t^{i,b}) \mathbf{1}_{\{x_1 \leq 0, x_2 \leq 0\}}) \leq 0.$$

Consequently, using Proposition 5.3 and the comparison theorem for BSDEs, we obtain the desired inequality (5.11).

### 5.2.2 Contracts with Monotone Cash Flows

If  $x_1 x_2 < 0$  then, from the proof of Proposition 5.4, we know that for some contracts  $(A, C)$  we have  $P_t^c(x_2, -A, -C) \geq P_t^h(x_1, A, C)$  for some  $\hat{t} \in [0, T]$ . The next theorem show that, for some special classes of contracts  $(A, C)$ , the inequality  $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$  is satisfied for all  $t \in [0, T]$ .

**Theorem 5.1** *Let either Assumption 5.1 or Assumption 5.2 be valid. If  $x_1 \geq 0, x_2 \leq 0$ , then for an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and such that the process  $A^C$  is decreasing on  $(0, T]$  we have, for every  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.},$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

*Proof.* From Proposition 5.3 and the inequalities  $x_1 \geq 0, x_2 \leq 0$ , we know that for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  we have  $P^h(x_1, A, C) = \tilde{Y}^{h,l,x_1} - C$  and  $P^c(x_2, -A, -C) = \tilde{Y}^{c,b,x_2} - C$  where  $(\tilde{Y}^{h,l,x_1}, \tilde{Z}^{h,l,x_1})$  is the unique solution of BSDE (6.2) and  $(\tilde{Y}^{c,b,x_2}, \tilde{Z}^{c,b,x_2})$  is the unique solution of BSDE (6.3). Since

$$g^{h,l}(t, x_1, 0, 0) = g^{c,b}(t, x_2, 0, 0) = 0,$$

and  $A^C$  is a decreasing process then, from the comparison theorem for BSDEs, we have  $\tilde{Y}^{h,l,x_1} \geq 0$  and  $\tilde{Y}^{c,b,x_2} \geq 0$ . Since  $x_1 \geq 0$ , BSDE (5.7) becomes

$$\begin{cases} d\tilde{Y}_t^{h,l,x_1} = \tilde{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{\text{cld}} + \tilde{g}^{h,l}(t, x_1, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,l,x_1} = 0, \end{cases} \quad (5.12)$$

where the generator  $\tilde{g}^{h,l}(t, x, y, z)$  does not depend on  $x$  and it is given by (recall that  $\bar{z}_t^i = z^i S_t^i$ )

$$\tilde{g}^{h,l}(t, x, y, z) := \sum_{i=1}^d \beta_t^i \bar{z}_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{z}_t^i)^+ + r_t^l y + r_t^l \sum_{i=1}^d (\bar{z}_t^i)^-.$$

Since

$$\begin{aligned} g^{c,b}(t, x, y, z) &= \sum_{i=1}^d \beta_t^i \bar{z}_t^i + \sum_{i=1}^d r_t^{i,b} (-\bar{z}_t^i)^+ + x r_t^b B_t^b \\ &\quad - r_t^l \left( -y + x B_t^b + \sum_{i=1}^d (-\bar{z}_t^i)^- \right)^+ + r_t^b \left( -y + x B_t^b + \sum_{i=1}^d (-\bar{z}_t^i)^- \right)^- \\ &\geq \sum_{i=1}^d \beta_t^i \bar{z}_t^i + \sum_{i=1}^d r_t^{i,b} (-\bar{z}_t^i)^+ + x r_t^b B_t^b - r_t^b \left( -y + x B_t^b + \sum_{i=1}^d (-\bar{z}_t^i)^- \right) \\ &= \sum_{i=1}^d \beta_t^i \bar{z}_t^i + \sum_{i=1}^d r_t^{i,b} (-\bar{z}_t^i)^+ + r_t^b y - r_t^b \sum_{i=1}^d (-\bar{z}_t^i)^-, \end{aligned}$$

we obtain

$$\begin{aligned} &\tilde{g}^{h,l}(t, x, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) - g^{c,b}(t, x, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) \\ &\leq (r_t^l - r_t^b) \tilde{Y}_t^{h,l,x_1} - \sum_{i=1}^d r_t^{i,b} |\tilde{Z}_t^{h,l,x_1,i} S_t^i| + r_t^l \sum_{i=1}^d (\tilde{Z}_t^{h,l,x_1,i} S_t^i)^- + r_t^b \sum_{i=1}^d (-\tilde{Z}_t^{h,l,x_1,i} S_t^i)^- \\ &= (r_t^l - r_t^b) \tilde{Y}_t^{h,l,x_1} + \sum_{i=1}^d (r_t^l - r_t^{i,b}) (\tilde{Z}_t^{h,l,x_1,i} S_t^i)^- + \sum_{i=1}^d (r_t^b - r_t^{i,b}) (-\tilde{Z}_t^{h,l,x_1,i} S_t^i)^- \leq 0. \end{aligned}$$

The comparison theorem for BSDEs gives  $\tilde{Y}_t^{h,l,x_1} \geq \tilde{Y}_t^{c,b,x_2}$  and thus  $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$ ,  $\tilde{\mathbb{P}}^\beta$ -a.s. for every  $t \in [0, T]$ .  $\square$

**Theorem 5.2** *Let either Assumption 5.1 or Assumption 5.2 be valid. If  $x_1 \leq 0$ ,  $x_2 \geq 0$ , then for an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and such that the process  $A^C$  is increasing on  $(0, T]$  we have, for every  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.},$$

so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.

*Proof.* From Proposition 5.3 and  $x_1 \leq 0$ ,  $x_2 \geq 0$ , we know that  $P^h(x_1, A, C) = \tilde{Y}^{h,b,x_1} - C$  and  $P^c(x_2, -A, -C) = \tilde{Y}^{c,l,x_2} - C$  where  $(\tilde{Y}^{h,b,x_1}, \tilde{Z}^{h,b,x_1})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{h,b,x_1} = \tilde{Z}_t^{h,b,x_1,*} d\tilde{S}_t^{\text{cld}} + g^{h,b}(t, x_1, \tilde{Y}_t^{h,b,x_1}, \tilde{Z}_t^{h,b,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,b,x_1} = 0, \end{cases} \quad (5.13)$$

where

$$g^{h,b}(t, x, y, z) := \sum_{i=1}^d z_t^i \beta_t^i S_t^i - x r_t^b B_t^b + g(t, y + x B_t^b, z).$$

and  $(\tilde{Y}^{c,l,x_2}, \tilde{Z}^{c,l,x_2})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{c,l,x_2} = \tilde{Z}_t^{c,l,x_2,*} d\tilde{S}_t^{\text{cld}} + g^{c,l}(t, x_2, \tilde{Y}_t^{c,l,x_2}, \tilde{Z}_t^{c,l,x_2}) dt + dA_t^C, \\ \tilde{Y}_T^{c,l,x_2} = 0, \end{cases} \quad (5.14)$$

where

$$g^{c,l}(t, x, y, z) := \sum_{i=1}^d z_t^i \beta_t^i S_t^i + x r_t^l B_t^l - g(t, -y + x B_t^l, -z).$$

Since

$$g^{h,b}(t, x_1, 0, 0) = g^{c,l}(t, x_2, 0, 0) = 0,$$

and the process  $A^C$  is assumed to be increasing, from Theorem 3.3 in [14], we obtain  $\tilde{Y}^{h,b,x_1} \leq 0$  and  $\tilde{Y}^{c,l,x_2} \leq 0$ . Therefore, since  $x_2 \geq 0$ , we see that  $g^{c,l}(t, x, y, z)$  does not depend on  $x$  and

$$g^{c,l}(t, x, y, z) = \tilde{g}^{c,l}(t, x, y, z) := \sum_{i=1}^d \beta_t^i \bar{z}_t^i + \sum_{i=1}^d r_t^{i,b} (-\bar{z}_t^i)^+ + r_t^l y - r_t^l \sum_{i=1}^d (-\bar{z}_t^i)^-$$

where, as usual, we denote  $\bar{z}_t^i = z^i S_t^i$ . Furthermore, the function  $g^{h,b}(t, x, y, z)$  satisfies

$$\begin{aligned} g^{h,b}(t, x, y, z) &= \sum_{i=1}^d \beta_t^i \bar{z}_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{z}_t^i)^+ - x r_t^b B_t^b \\ &\quad + r_t^l \left( y + x B_t^b + \sum_{i=1}^d (\bar{z}_t^i)^- \right)^+ - r_t^b \left( y + x B_t^b + \sum_{i=1}^d (\bar{z}_t^i)^- \right)^- \\ &\leq \sum_{i=1}^d z_t^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{z}_t^i)^+ - x r_t^b B_t^b + r_t^b \left( y + x B_t^b + \sum_{i=1}^d (\bar{z}_t^i)^- \right) \\ &= \sum_{i=1}^d \beta_t^i \bar{z}_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{z}_t^i)^+ + r_t^b y + r_t^b \sum_{i=1}^d (-\bar{z}_t^i)^- \end{aligned}$$

and thus

$$\begin{aligned} &g^{h,b}(t, x, \tilde{Y}_t^{h,b,x_1}, \tilde{Z}_t^{h,b,x_1}) - \tilde{g}^{c,l}(t, x, \tilde{Y}_t^{h,b,x_1}, \tilde{Z}_t^{h,b,x_1}) \\ &\leq (r_t^b - r_t^l) \tilde{Y}_t^{h,b,x_1} - \sum_{i=1}^d r_t^{i,b} |\tilde{Z}_t^{h,b,x_1,i} S_t^i| + r_t^l \sum_{i=1}^d (-\tilde{Z}_t^{h,b,x_1,i} S_t^i)^- + r_t^b \sum_{i=1}^d (\tilde{Z}_t^{h,b,x_1,i} S_t^i)^- \\ &= (r_t^b - r_t^l) \tilde{Y}_t^{h,b,x_1} + \sum_{i=1}^d (r_t^l - r_t^{i,b}) (-\tilde{Z}_t^{h,b,x_1,i} S_t^i)^- + \sum_{i=1}^d (r_t^b - r_t^{i,b}) (\tilde{Z}_t^{h,b,x_1,i} S_t^i)^- \leq 0. \end{aligned}$$

The comparison theorem for BSDEs gives  $\tilde{Y}^{h,b,x_1} \geq \tilde{Y}^{c,l,x_2}$  and thus  $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$   $\tilde{\mathbb{P}}^\beta$ -a.s., for every  $t \in [0, T]$ .  $\square$

**Remark 5.4** Consider a contract  $(A, C)$  such that  $A^C$  is a decreasing process on  $(0, T]$ . If  $x_1 \geq 0$  then, from the proof of the proposition, we see that  $P^h(x_1, A, C)$  does not depend on the initial wealth  $x_1$ , that is, for every  $x, y \in \mathbb{R}_+$  we have  $P^h(x, A, C) = P^h(y, A, C)$ . This follows from the equality  $P^h(x_1, A, C) = \tilde{Y}^{h,l,x_1} - C$ , where  $(\tilde{Y}^{h,l,x_1}, \tilde{Z}^{h,l,x_1})$  is the unique solution of BSDE (5.12), which is independent of  $x_1$ . Note that the above relation hinges on the condition  $x_1 \geq 0$ . Indeed, when  $x_1 \leq 0$ , then  $P^h(x_1, A, C)$  does not enjoy the independence property. Furthermore, for any  $x_2 \in \mathbb{R}$ , the price  $P^c(x_2, -A, -C)$  does not have such property. Finally, for a contract  $(A, C)$  such that  $A^C$  is an increasing process on  $(0, T]$ , if  $x_2 \geq 0$ , then  $P^c(x_2, -A, -C)$  does not depend on the initial wealth  $x_2$ , but  $P^h(x_1, A, C)$  does not have this property.

The above-mentioned property is intuitively clear from its financial interpretation. In essence, the independence of the hedger's price of his non-negative positive wealth is a consequence of the last constraint in equation (2.8), which states that the hedger cannot use his initial endowment to buy shares for the purpose of hedging. Of course, when he sells shares to replicate an option, as is the case for the put option, then, obviously, the fact that his initial endowment is positive is also irrelevant.

**Remark 5.5** Assume that  $x_1 > 0$  and  $x_2 < 0$ . We claim that if  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ , then we can find a date  $\hat{t} \in [0, T]$  and a contract  $(A, C)$  with an increasing process  $A^C$  such that

$$P_{\hat{t}}^c(x_2, -A, -C) > P_{\hat{t}}^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

To this end, it suffices consider a contract  $(A, C)$  with  $C = 0$  and  $A_t = p \mathbf{1}_{[0, T]}(t) + \alpha \mathbf{1}_{[t_0, T]}(t)$  where  $t_0 \in (0, T)$ ,  $r_t \in (r_t^l, r_t^b)$  for every  $t \in [0, T]$  and  $\alpha$  satisfies

$$0 < \alpha \leq \min \{x_1 B_{t_0}^l, -x_2 B_{t_0}^b\}.$$

We set  $x = x_1 - \alpha (B_{t_0}^l)^{-1} \geq 0$  and we define the strategy  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where  $\xi^i = \psi^{i,b} = \psi^b = \eta^b = \eta^l = 0$  for all  $i = 1, 2, \dots, d$  and

$$\psi_t^l = x \mathbf{1}_{[0, t_0)} + (B_{t_0}^l)^{-1} (x B_{t_0}^l + \alpha) \mathbf{1}_{[t_0, T]}.$$

Then we have

$$V_T(x, \varphi, A, C) = x B_T^l + \alpha e^{\int_{t_0}^T r_u^l du} = (x_1 - \alpha (B_{t_0}^l)^{-1}) B_T^l + \alpha e^{\int_{t_0}^T r_u^l du} = x_1 (B_T^l) = V_T^0(x_1).$$

Hence the hedger's self-financing strategy  $(x, \varphi, A, C)$  replicates the contract  $(A, C)$  on  $[0, T]$  and, in fact, this is the unique replicating strategy. From Definition 3.7, it follows that  $P_0^h(x_1, A, C) = x - x_1 = -\alpha(B_{t_0}^l)^{-1}$ . Let us now consider the contract from the perspective of the counterparty. For  $\tilde{x} = x_2 + \alpha(B_{t_0}^b)^{-1} \leq 0$ , we define the strategy  $\tilde{\varphi} = (\tilde{\xi}^1, \dots, \tilde{\xi}^d, \tilde{\psi}^l, \tilde{\psi}^b, \tilde{\psi}^{1,b}, \dots, \tilde{\psi}^{d,b}, \tilde{\eta}^b, \tilde{\eta}^l)$  where  $\tilde{\xi}^i = \tilde{\psi}^{i,b} = \tilde{\psi}^l = \tilde{\eta}^b = \tilde{\eta}^l = 0$  for all  $i = 1, 2, \dots, d$  and

$$\tilde{\psi}_t^b = \tilde{x} \mathbf{1}_{[0, t_0)} + (B_{t_0}^b)^{-1} (\tilde{x} B_{t_0}^b + \alpha) \mathbf{1}_{[t_0, T]}.$$

Then we have

$$V_T(\tilde{x}, \tilde{\varphi}, A, C) = \tilde{x} B_T^b + \alpha e^{\int_{t_0}^T r_u^b du} = x_2 B_T^b = V_T^0(x_2).$$

Therefore, the self-financing strategy  $(\tilde{x}, \tilde{\varphi}, -A, -C)$  is the unique replicating strategy for the contract  $(-A, -C)$  on  $[0, T]$  and, from Definition 3.8, it follows that  $P_0^c(x_2, -A, -C) = x_2 - \tilde{x} = -\alpha(B_{t_0}^b)^{-1}$ . Moreover, since  $r^l < r^b$  and  $\alpha > 0$ , we have that

$$P_0^h(x_1, A, C) = -\alpha(B_{t_0}^l)^{-1} < -\alpha(B_{t_0}^b)^{-1} = P_0^c(x_2, -A, -C).$$

Consequently, under the assumption that  $r^l < r^b$  we have found a contract  $(A, C)$  and a date  $\hat{t} = 0$  such that  $P_0^c(x_2, -A, -C) > P_0^h(x_1, A, C)$ . This means that the range of bilaterally profitable prices  $\mathcal{R}_0^p(x_1, x_2)$  for  $(A, C)$  is non-empty.

### 5.3 Monotonicity of Prices with Respect to the Initial Endowment

As shown in the preceding subsection, the initial endowment plays an important role in the pricing inequality. In the following, we examine in more details the impact of the initial endowment on the ex-dividend price. In view of Remark 5.3, we only need to work under Assumption 5.1.

**Proposition 5.6** *Let either Assumption 5.1 or Assumption 5.2 be valid and let a contract  $(A, C)$  be admissible under  $\tilde{\mathbb{P}}^\beta$ . Then the hedger's price satisfies:*

(i) if  $\bar{x} \geq x \geq 0$ , then

$$P_t^h(\bar{x}, A, C) \leq P_t^h(x, A, C), \quad (5.15)$$

(ii) if  $0 \geq \bar{x} \geq x$ , then

$$P_t^h(\bar{x}, A, C) \geq P_t^h(x, A, C), \quad (5.16)$$

and the counterparty's price satisfies:

(i) if  $\bar{x} \geq x \geq 0$ , then

$$P_t^c(\bar{x}, -A, -C) \geq P_t^c(x, -A, -C), \quad (5.17)$$

(ii) if  $0 \geq \bar{x} \geq x$ , then

$$P_t^c(\bar{x}, -A, -C) \leq P_t^c(x, -A, -C). \quad (5.18)$$

*Proof.* Let us denote

$$g^{l,h}(x) := -xr_t^l B_t^l + g(t, y + xB_t^l, z), \quad g^{b,h}(x) := -xr_t^b B_t^b + g(t, y + xB_t^b, z),$$

and

$$g^{l,c}(x) := xr_t^l B_t^l - g(t, -y + xB_t^l, -z), \quad g^{b,c}(x) := xr_t^b B_t^b - g(t, -y + xB_t^b, -z),$$

where (see (5.3))

$$g(t, y, z) = -\sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^-.$$

If we denote  $K := y + \sum_{i=1}^d (z^i S_t^i)^-$  and  $\tilde{K} := -y + \sum_{i=1}^d (-z^i S_t^i)^-$ , then

$$\begin{aligned} g^{l,h}(x) &= -xr_t^l B_t^l + r_t^l (xB_t^l + K)^+ - r_t^b (xB_t^l + K)^- \\ &= -r_t^l (xB_t^l + K) + r_t^l (xB_t^l + K)^+ - r_t^b (xB_t^l + K)^- + r_t^l K \\ &= r_t^l (xB_t^l + K)^- - r_t^b (xB_t^l + K)^- + r_t^l K \\ &= (r_t^l - r_t^b) (xB_t^l + K)^- + r_t^l K \end{aligned}$$



and

$$\begin{aligned} g^{b,h}(x) &= -xr_t^b B_t^l + r_t^l(xB_t^b + K)^+ - r_t^b(xB_t^b + K)^- \\ &= -r_t^b(xB_t^b + K) + r_t^l(xB_t^b + K)^+ - r_t^b(xB_t^b + K)^- + r_t^b K \\ &= (r_t^l - r_t^b)(xB_t^l + K)^+ + r_t^b K. \end{aligned}$$

Similarly,

$$g^{l,c}(x) = (r_t^b - r_t^l)(xB_t^l + \tilde{K})^- - r_t^l \tilde{K}$$

and

$$g^{b,c}(x) = (r_t^b - r_t^l)(xB_t^l + K)^+ - r_t^b K.$$

Therefore, the functions  $\tilde{g}^{l,h}(x)$  and  $\tilde{g}^{b,c}(x)$  are increasing with respect to  $x$ , whereas the functions  $\tilde{g}^{b,h}(x)$  and  $\tilde{g}^{l,c}(x)$  are decreasing with respect to  $x$ . Consequently, from the comparison theorem for BSDEs, if  $\bar{x} \geq x \geq 0$ , then  $\tilde{Y}^{h,l,x} \leq \tilde{Y}^{h,l,\bar{x}}$  where  $(\tilde{Y}^{h,l,x}, \tilde{Z}^{h,l,x})$  is the unique solution of BSDE (6.2). Moreover,  $\tilde{Y}^{c,l,x} \geq \tilde{Y}^{c,l,\bar{x}}$  where  $(\tilde{Y}^{c,l,x}, \tilde{Z}^{c,l,x})$  is the unique solution of BSDE (5.14). Then from Remark 5.3, we deduce that (5.15) and (5.17) hold. For  $0 \geq \bar{x} \geq x$  one can show, using similar arguments, that (5.16) and (5.18) are valid.  $\square$

By combining Propositions 5.2–5.6, we obtain the following result, which summarizes the properties of unilateral prices.

**Theorem 5.3** *Let either Assumption 5.1 or Assumption 5.2 be valid. Then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  the following statements are valid:*

(i) *if  $\bar{x} \geq x \geq 0$ , then for all  $t \in [0, T]$*

$$P_t^c(x, -A, -C) \leq P_t^c(\bar{x}, -A, -C) \leq P_t^h(\bar{x}, A, C) \leq P_t^h(x, A, C). \quad (5.19)$$

(ii) *if  $0 \geq \bar{x} \geq x$ , then for all  $t \in [0, T]$*

$$P_t^h(\bar{x}, A, C) \geq P_t^h(x, A, C) \geq P_t^c(x, -A, -C) \geq P_t^c(\bar{x}, -A, -C). \quad (5.20)$$

Moreover, if  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ , then for  $\bar{x} > 0 > x$ , there exists  $(\hat{t}, A, C)$  such that

$$P_{\hat{t}}^c(x, -A, -C) > P_{\hat{t}}^h(\bar{x}, A, C) \geq P_{\hat{t}}^c(\bar{x}, -A, -C)$$

and there also exists  $(\hat{t}, A, C)$  such that

$$P_{\hat{t}}^h(\bar{x}, A, C) \geq P_{\hat{t}}^c(\bar{x}, -A, -C) > P_{\hat{t}}^h(x, A, C).$$

**Corollary 5.1** *Under the assumptions of Proposition 5.6, for any contract  $(A, C)$  and any date  $t \in [0, T]$*

$$P_t^c(0, -A, -C) \leq P_t^c(x, -A, -C) \leq P_t^h(x, A, C) \leq P_t^h(0, A, C), \quad (5.21)$$

so that  $\mathcal{R}_t^f(x, x) \subset \mathcal{R}_t^f(0, 0)$ .

The above corollary shows that an investor with either a positive or a negative initial endowment has a potential advantage over an investor with null initial wealth to enter any contract  $(A, C)$  at any time  $t$ . This conclusion is plausible, since the borrowing rate is higher than the lending rate.

Indeed, for the same strategy, when an investor who has zero initial endowment needs to borrow money in order to hedge a contract, an investor with a positive initial endowment may use money from his initial wealth for the same purpose. Similarly, when an investor with null initial endowment needs to lend money in order to implement his hedging strategy, an investor with a negative initial endowment can use instead a surplus of cash to repay his debt. These features create a comparative advantage.

Using Corollary 5.1 and Proposition 5.6, we can examine the asymptotic properties of  $P_t^h(x, A, C)$  and  $P_t^c(x, -A, -C)$  when the initial endowment  $x$  tends to either  $\infty$  or  $-\infty$ .

**Proposition 5.7** *Let the assumptions of Proposition 5.6 be valid. For any contract  $(A, C)$  and any date  $t \in [0, T]$ , there exist  $\mathbb{G}$ -adapted processes, denoted by  $P_t^{h,A,C,+}$ ,  $P_t^{h,A,C,-}$ ,  $P_t^{c,-A,-C,+}$  and  $P_t^{c,-A,-C,-}$ , such that*

$$P_t^{h,A,C,+}, P_t^{h,A,C,-}, P_t^{c,-A,-C,+}, P_t^{c,-A,-C,-} \in [P_t^c(0, -A, -C), P_t^h(0, A, C)] = \mathcal{R}_0^f(0, 0)$$

and

$$\begin{aligned} \lim_{x \rightarrow +\infty} P_t^h(x, A, C) &= P_t^{h,A,C,+} \geq P_t^{c,-A,-C,+} = \lim_{x \rightarrow +\infty} P_t^c(x, -A, -C), \\ \lim_{x \rightarrow -\infty} P_t^h(x, A, C) &= P_t^{h,A,C,-} \geq P_t^{c,-A,-C,-} = \lim_{x \rightarrow -\infty} P_t^c(x, -A, -C). \end{aligned}$$

*Proof.* The statement easily follows from Proposition 5.6 and Corollary 5.1.  $\square$

We can only have  $P_t^{h,A,C,+} \geq P_t^{c,A,C,+}$  and  $P_t^{h,A,C,-} \geq P_t^{c,A,C,-}$ . Other comparison results between these four processes are still unclear. Indeed, if  $r^l$  and  $r^b$  are deterministic and such that  $r_t^l < r_t^b$  for every  $t \in [0, T]$ , then there exists  $(\hat{t}, A, C)$  such that

$$P_{\hat{t}}^{h,A,C,+} \geq P_{\hat{t}}^{c,A,C,+} > P_{\hat{t}}^{h,A,C,-} \geq P_{\hat{t}}^{c,A,C,-},$$

as well as there exists  $(\hat{t}, A, C)$  such that

$$P_{\hat{t}}^{h,A,C,-} \geq P_{\hat{t}}^{c,A,C,-} > P_{\hat{t}}^{h,A,C,+} \geq P_{\hat{t}}^{c,A,C,+}.$$

Now, we consider a special case of a contract  $(A, C)$  and  $t \in [0, T]$  such that

$$P_t^{h,A,C} := \min \left\{ P_t^{h,A,C,+}, P_t^{h,A,C,-} \right\} \geq \max \left\{ P_t^{c,-A,-C,+}, P_t^{c,-A,-C,-} \right\} =: P_t^{c,-A,-C}.$$

Then  $[P_t^{c,-A,-C}, P_t^{h,A,C}]$  is the bilateral fair pricing range for all investors with identical, but otherwise arbitrary, initial endowment, meaning that

$$[P_t^{c,-A,-C}, P_t^{h,A,C}] = \bigcap_{x \in \mathbb{R}} [P_t^c(x, -A, -C), P_t^h(x, A, C)] = \bigcap_{x \in \mathbb{R}} \mathcal{R}_t^f(x, x).$$

The following stability of unilateral ex-dividend prices with respect to the initial endowment can also be established using Proposition 3.1 in [14]. For the reader's convenience, we recall two alternative versions of the Lipschitz condition, which were employed in [14] (see Definitions 2.1 and 3.1 in [14]). Let  $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{G} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function such that  $h(\cdot, \cdot, y, z)$  is a  $\mathbb{G}$ -adapted process for any fixed  $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ , and let  $m$  be the process introduced in either Assumption 5.1 or Assumption 5.2.

**Definition 5.2** We say that  $h$  satisfies the *uniform Lipschitz condition* if there exists a constant  $L$  such that, for all  $t \in [0, T]$  and  $y_1, y_2 \in \mathbb{R}$ ,  $z_1, z_2 \in \mathbb{R}^d$ ,

$$|h(t, y_1, z_1) - h(t, y_2, z_2)| \leq L(|y_1 - y_2| + \|z_1 - z_2\|), \quad \mathbb{P} - \text{a.s.} \quad (5.22)$$

We say that  $h$  satisfies the *m-Lipschitz condition* if there exist two strictly positive and  $\mathbb{G}$ -adapted processes  $\rho$  and  $\theta$  such that, for all  $t \in [0, T]$  and  $y_1, y_2 \in \mathbb{R}$ ,  $z_1, z_2 \in \mathbb{R}^d$ ,

$$|h(t, y_1, z_1) - h(t, y_2, z_2)| \leq \rho_t |y_1 - y_2| + \theta_t \|m_t^*(z_1 - z_2)\|. \quad (5.23)$$

**Theorem 5.4** *Let either Assumption 5.1 or Assumption 5.2 be valid. Then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ , there exists a constant  $K_0$  such that*

$$\mathbb{E}_{\mathbb{P}} \left[ \sup_{t \in [0, T]} |P_t^h(x_1, A, C) - P_t^h(x_2, A, C)| + \sup_{t \in [0, T]} |P_t^c(x_1, -A, -C) - P_t^c(x_2, -A, -C)| \right] \leq K_0 |x_1 - x_2|.$$

*Proof.* From Remark 5.3, we have  $P_t^h(x_i, A, C) = \tilde{Y}_t^{h,x_i} - C_t$  for every  $t \in [0, T)$ , where  $(\tilde{Y}^{h,x_i}, \tilde{Z}^{h,x_i})$  is the solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{h,x_i} = \tilde{Z}_t^{h,x_i,*} d\tilde{S}_t^{\text{cld}} + g^h(t, x_i, \tilde{Y}_t^{h,x_i}, \tilde{Z}_t^{h,x_i,*}) dt + dA_t^C, \\ \tilde{Y}_T^{h,x_i} = 0, \end{cases} \quad (5.24)$$

where

$$g^h(t, x, y, z) := \sum_{i=1}^d \beta_t^i z_t^i S_t^i + (g(t, y + xB_t^l, z) - xr_t^l B_t^l) \mathbf{1}_{\{x \geq 0\}} + (g(t, y + xB_t^b, z) - xr_t^b B_t^b) \mathbf{1}_{\{x \leq 0\}}.$$

It is not hard to check that if  $x_1 x_2 \geq 0$ , then there exists a constant  $K$ , which only depends on the bound for  $r^l$  and  $r^b$ , such that

$$|g^h(t, x_1, y, z) - g^h(t, x_2, y, z)| \leq K|x_1 - x_2|.$$

Consequently, if  $x_1 x_2 < 0$ , then

$$\begin{aligned} |g^h(t, x_1, y, z) - g^h(t, x_2, y, z)| &\leq |g^h(t, x_1, y, z) - g^h(t, 0, y, z)| + |g^h(t, 0, y, z) - g^h(t, x_2, y, z)| \\ &\leq K|x_1| + K|x_2| = K|x_1 - x_2|. \end{aligned}$$

We conclude that there exists a constant  $K$ , which depends only on the bound for  $r^l$  and  $r^b$ , such that

$$|g^h(t, x_1, y, z) - g^h(t, x_2, y, z)| \leq K|x_1 - x_2|, \text{ for all } x_1, x_2 \in \mathbb{R}.$$

Under Assumption 5.1 (resp., Assumption 5.2), for a fixed  $x \in \mathbb{R}$ ,  $g^h(t, x, y, z)$  satisfies (5.23) with  $\rho = \theta = \hat{L}$ , where a constant  $\hat{L}$  depends on the bound for  $r^l, r^b, r^{i,b}$  and  $S^i$  for  $i = 1, 2, \dots, d$ , as well as the lower bound for  $|m|$  (resp., a constant  $\hat{L}$  depends on the bound  $r^l, r^b$  and  $r^{i,b}$ ). Consequently, there always exists a constant  $\hat{L}$ , such that the driver satisfies the Lipschitz condition (5.23) with processes  $\rho = \theta = \hat{L}$ . Consequently, as in Section 3.2 in [14], we deduce that the spaces  $\hat{\mathcal{H}}_\lambda^2$  and  $\hat{\mathcal{H}}_0^2$ , (resp., the spaces  $\hat{L}_\lambda^2$  and  $\hat{L}_0^2$ ) may be identified, since the related norms are equivalent. Moreover, one can check  $\alpha^{-1}g^h(t, x, 0, 0) \in \hat{\mathcal{H}}_0^2$ .

By an application of Proposition 3.2 in [14], there exists a constant  $K_0$  such that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \sup_{t \in [0, T]} |P_t^h(x_1, A, C) - P_t^h(x_2, A, C)|^2 \right] &= \mathbb{E}_{\mathbb{P}} \left[ \sup_{t \in [0, T]} |\tilde{Y}_t^{h,x_1} - \tilde{Y}_t^{h,x_2}|^2 \right] \\ &\leq K_0 \left| \alpha^{-1}g^h(t, x_1, \tilde{Y}_t^{h,x_2} - A_t^C, \tilde{Z}_t^{h,x_2}) - \alpha^{-1}g^h(t, x_2, \tilde{Y}_t^{h,x_2} - A_t^C, \tilde{Z}_t^{h,x_2}) \right|_{\hat{\mathcal{H}}_0^2}^2 \\ &\leq K_0|x_1 - x_2|^2. \end{aligned}$$

Similarly, one can check that the same inequality holds for the counterparty's price.  $\square$

## 5.4 Price Independence of the Initial Endowment

We will now show that for a certain class of contracts the price is independent of the initial endowment. It is worth noting that an analogous result does not hold in Bergman's model studied in [15].

**Proposition 5.8** *Let  $x_1 \geq 0$  and either Assumption 5.1 or Assumption 5.2 be valid. Consider an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ . If the process  $A^C - A_0^C$  is decreasing, then the price  $P_t^h(x_1, A, C)$  is independent of  $x_1$ , so that  $P_t^h(x_1, A, C) = P_t^h(0, A, C)$  for all  $x_1 \geq 0$ .*

*Proof.* Since  $x_1 \geq 0$ , it follows from Proposition 5.3 that the hedger's price of any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  satisfies  $P_t^h(x_1, A, C) = \tilde{Y}_t^{h,l,x_1} - C_t$  where  $(\tilde{Y}^{h,l,x_1}, \tilde{Z}^{h,l,x_1})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{h,l,x_1} = \tilde{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{\text{cld}} + g^{h,l}(t, x_1, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,l,x_1} = 0, \end{cases} \quad (5.25)$$

where

$$g^{h,l}(t, x_1, y, z) := \sum_{i=1}^d \beta_t^i z^i S_t^i - x_1 r_t^l B_t^l - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ \\ + r_t^l \left( y + x_1 B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + x_1 B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^-.$$

Since  $g^{h,l}(t, x_1, 0, 0) = 0$  and the process  $A^C - A_0^C$  is decreasing, we deduce from the comparison theorem for BSDEs (see, for instance, Theorem 3.3 in [14] with  $U^1 = A^C - A_0^C$  and  $U^2 = 0$ ) that  $\tilde{Y}^{h,l,x_1} \geq 0$ . Since  $x_1 \geq 0$ , BSDE (5.25) can thus be represented as follows

$$\begin{cases} d\tilde{Y}_t^{h,l,x_1} = \tilde{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{\text{cld}} + \tilde{g}^{h,l}(t, x_1, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,l,x_1} = 0, \end{cases} \quad (5.26)$$

where the generator  $\tilde{g}^{h,l}(t, x_1, y, z)$  is independent of  $x_1$  and equals (recall that  $\tilde{z}_t^i = z^i S_t^i$ )

$$\tilde{g}^{h,l}(t, x, y, z) := \sum_{i=1}^d \beta_t^i \tilde{z}_t^i - \sum_{i=1}^d r_t^{i,b} (\tilde{z}_t^i)^+ + r_t^l y + r_t^l \sum_{i=1}^d (\tilde{z}_t^i)^-.$$

Obviously, the unique solution to BSDE (5.26) is independent of  $x_1$  and thus the price  $P_t^h(x_1, A, C) = \tilde{Y}_t^{h,l,x_1} - C_t$  enjoys the same property.  $\square$

## 5.5 Positive Homogeneity of the Hedger's Price

We consider once again the hedger's price and we show that it is positively homogeneous with respect to the size of the contract and the non-negative initial endowment. Observe that this property is no longer true if only the size of the contract (but not the initial endowment) is scaled by a non-negative number  $\lambda$  (of course, unless the price is independent of the initial endowment, as in Proposition 5.8).

**Proposition 5.9** *Let  $x_1 \geq 0$  and either Assumption 5.1 or Assumption 5.2 be valid. Consider an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ . If the process  $C \in \tilde{\mathcal{H}}_0^2$ , then for all  $\lambda \in \mathbb{R}_+$*

$$P_t^h(\lambda x_1, \lambda A, \lambda C) = \lambda P_t^h(x_1, A, C). \quad (5.27)$$

*Proof.* It is obvious that (5.27) holds for  $\lambda = 0$ . Suppose that  $\lambda > 0$ . Once again, from Proposition 5.3, we know that  $P^h(x_1, A, C) = \tilde{Y}^{h,l,x_1} - C$  where  $(\tilde{Y}^{h,l,x_1}, \tilde{Z}^{h,l,x_1})$  is the unique solution to (5.25). Moreover,  $P^h(\lambda x_1, \lambda A, \lambda C) = \tilde{Y}^{h,l,\lambda x_1} - \lambda C$  where  $(\tilde{Y}^{h,l,\lambda x_1}, \tilde{Z}^{h,l,\lambda x_1})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{h,l,\lambda x_1} = \tilde{Z}_t^{h,l,\lambda x_1,*} d\tilde{S}_t^{\text{cld}} + g^{h,l}(t, \lambda x_1, \tilde{Y}_t^{h,l,\lambda x_1}, \tilde{Z}_t^{h,l,\lambda x_1}) dt + \lambda dA_t^C, \\ \tilde{Y}_T^{h,l,\lambda x_1} = 0. \end{cases}$$

Recall that  $A^C = A + C + F^C$  where  $F_t^C := -\int_0^t r_u^c C_u du$ . Then  $P^h(x_1, A, C) = Y^1$  where  $(Y^1, Z^1)$  is the unique solution of the following BSDE (since  $(A, C)$  is admissible and  $C \in \tilde{\mathcal{H}}_0^2$ , the well-posedness of this BSDE is easy to check)

$$\begin{cases} dY_t^1 = Z_t^{1,*} d\tilde{S}_t^{\text{cld}} + g^{h,l}(t, x_1, Y_t^1 + C_t, Z_t^1) dt + d(A_t + F_t^C), \\ Y_T^1 = 0. \end{cases}$$

Similarly,  $P^h(\lambda x_1, \lambda A, \lambda C) = Y^2$  where  $(Y^2, Z^2)$  is the unique solution of the following BSDE

$$\begin{cases} dY_t^2 = Z_t^{2,*} d\tilde{S}_t^{\text{cld}} + g^{h,l}(t, \lambda x_1, Y_t^2 + \lambda C_t, Z_t^2) dt + \lambda d(A_t + F_t^C), \\ Y_T^2 = 0. \end{cases} \quad (5.28)$$

For  $Y := \lambda Y^1$  and  $Z = \lambda Z^1$ , we have

$$\begin{cases} dY_t = Z_t^* d\tilde{S}_t^{\text{cld}} + \lambda g^{h,l}(t, x_1, \lambda^{-1}Y_t + C_t, \lambda^{-1}Z_t) dt + \lambda d(A_t + F_t^C), \\ Y_T = 0. \end{cases} \quad (5.29)$$

Hence to complete the proof, it suffices to observe that the equality

$$\lambda g^{h,l}(t, x_1, \lambda^{-1}y + C_t, \lambda^{-1}z) = g^{h,l}(t, \lambda x_1, y + \lambda C_t, z)$$

is satisfied for all  $\lambda > 0$ .  $\square$

## 5.6 European Claims and Related Pricing PDEs

For simplicity of presentation, we assume that  $d = 1$ , so that there is only one risky asset  $S = S^1$ . It is clear, however, that the results obtained in this subsection can be easily extended to a multi-asset case. Moreover, we postulate that the interest rates  $r^l$  and  $r^b$  are deterministic. We examine valuation and hedging of an uncollateralized European contingent claim starting from a fixed time  $t \in [0, T]$ , that is, we set  $C = 0$ . A generic path-independent claim of European style pays a single cash flow  $H(S_T)$  on the expiration date  $T > 0$ , so that

$$A_t - A_0 = -H(S_T)\mathbf{1}_{[T,T]}(t).$$

For any fixed  $t < T$ , the risky asset  $S$  has the ex-dividend price dynamics under  $\mathbb{P}$  given by the following expression, for  $u \in [t, T]$ ,

$$dS_u = \mu(u, S_u) du + \sigma(u, S_u) dW_u, \quad S_t = s \in \mathcal{O}, \quad (5.30)$$

where  $W$  is a one-dimensional Brownian motion and  $\mathcal{O}$  is the domain of real values that are attainable by the diffusion process  $S$  (usually  $\mathcal{O} = \mathbb{R}_+$ ). Moreover, the coefficients  $\mu$  and  $\sigma$  are such that SDE (5.30) has a unique strong solution. We also assume that the volatility coefficient  $\sigma$  is bounded and bounded away from zero. Finally, the dividend process equals  $A_t^1 = \int_0^t \kappa(u, S_u) du$ .

Our first goal is to derive the hedger's pricing PDE for a path-independent European claim. We observe that

$$d\tilde{S}_u^{\text{cld}} = dS_u + dA_u^1 - \beta(u, S_u) du = (\mu(u, S_u) + \kappa(u, S_u) - \beta(u, S_u)) du + \sigma(u, S_u) dW_u.$$

From the Girsanov theorem, if we denote

$$a_u := (\sigma(u, S_u))^{-1}(\mu(u, S_u) + \kappa(u, S_u) - \beta(u, S_u))$$

and define the probability measure  $\tilde{\mathbb{P}}$  as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ - \int_t^T a_u dW_u - \frac{1}{2} \int_t^T |a_u|^2 du \right\},$$

then  $\tilde{\mathbb{P}}$  is equivalent to  $\mathbb{P}$  and the process  $\tilde{W}$  is the Brownian motion under  $\tilde{\mathbb{P}}$ , where  $d\tilde{W}_u := dW_u + a_u du$ . It is easy to see that

$$d\tilde{S}_u^{\text{cld}} = \sigma(u, S_u) d\tilde{W}_u$$

and thus we conclude that  $\tilde{S}^{\text{cld}}$  is a  $(\tilde{\mathbb{P}}, \mathbb{G})$ -martingale and  $\langle \tilde{S}^{\text{cld}} \rangle_u = \int_t^u |\sigma(v, S_v)|^2 dv$ . Therefore, either Assumption 5.1 or Assumption 5.2 holds, provided that we assume that the Brownian motion  $\tilde{W}$  has the PRP under  $(\mathbb{G}, \tilde{\mathbb{P}})$ . Of course, the latter assumption is not restrictive in the present set-up.

Since  $A$  has only a single cash flow at time  $T$  and  $C = 0$ , we deduce from Proposition 5.3 that, for any initial endowment  $x_1 \in \mathbb{R}$ , the hedger's prices satisfies  $P^h(x_1, A, C) = \tilde{Y}^{h,x_1}$ , where  $(\tilde{Y}^{h,x_1}, \tilde{Z}^{h,x_1})$  is the unique solution of following BSDE driven by the Brownian motion  $\tilde{W}$

$$\begin{cases} d\tilde{Y}_u^{h,x_1} = \tilde{Z}_u^{h,x_1} \sigma(u, S_u) d\tilde{W}_u + g^h(u, x_1, S_u, \tilde{Y}_u^{h,x_1}, \tilde{Z}_u^{h,x_1}) du, \\ \tilde{Y}_T^{h,x_1} = H(S_T), \end{cases} \quad (5.31)$$

where for  $x_1 \geq 0$

$$g^h(u, x_1, s, y, z) := z\beta(u, s) - x_1 r_u^l B_u^l - r_u^{1,b}(zs)^+ + r_u^l \left( y + x_1 B_u^l + (zs)^- \right)^+ - r_u^b \left( y + x_1 B_u^l + (zs)^- \right)^-$$

and for  $x_1 \leq 0$

$$g^h(u, x_1, s, y, z) := z\beta(u, s) - x_1 r_u^b B_u^b - r_u^{1,b}(zs)^+ + r_u^l \left( y + x_1 B_u^b + (zs)^- \right)^+ - r_u^b \left( y + x_1 B_u^b + (zs)^- \right)^-.$$

The well-posedness of BSDE (5.31) is well known under mild assumptions, since we assumed that  $\tilde{W}$  has the PRP under  $(\mathbb{G}, \tilde{\mathbb{P}})$ . The unique replicating strategy for the hedger equals  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  where  $\xi_u = \tilde{Z}_u^{h,x_1}$ ,  $\psi_u^{1,b} = -(B_u^{1,b})^{-1}(\xi_u S_u)^+$  and

$$\begin{aligned} \psi_u^l &= (B_u^l)^{-1} \left( \tilde{Y}_u^{h,x_1} + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} + (\xi_u S_u)^- \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( \tilde{Y}_u^{h,x_1} + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} + (\xi_u S_u)^- \right)^-. \end{aligned}$$

In the next step, we fix a date  $t \in [0, T]$  and we assume that  $S_t^{s,t} = s \in \mathcal{O}$ . Note that under  $\tilde{\mathbb{P}}$ , for all  $u \in [t, T]$ ,

$$dS_u^{s,t} = (\beta(u, S_u^{s,t}) - \kappa(u, S_u^{s,t})) du + \sigma(u, S_u^{s,t}) d\tilde{W}_u.$$

It is clear that the solution  $(\tilde{Y}^{h,x_1}, \tilde{Z}^{h,x_1})$  will now depend on the initial value  $s$  at time  $t$  of the stock price; to emphasize this feature, we write  $(\tilde{Y}^{h,x_1,s}, \tilde{Z}^{h,x_1,s})$ . Furthermore, if we set  $(Y_u^{h,x_1,s}, Z_u^{h,x_1,s}) := (\tilde{Y}_u^{h,x_1,s}, \tilde{Z}_u^{h,x_1,s} \sigma(u, S_u^{s,t}))$  and

$$\bar{g}^h(u, x_1, s, y, z) = g^h(u, x_1, s, y, z\sigma^{-1}(u, s)),$$

then BSDE (5.31) yields

$$\begin{cases} dY_u^{h,x_1,s} = Z_u^{h,x_1,s} d\tilde{W}_u + \bar{g}^h(u, x_1, S_u^{s,t}, Y_u^{h,x_1,s}, Z_u^{h,x_1,s}) du, \\ Y_T^{h,x_1,s} = H(S_T^{s,t}). \end{cases} \quad (5.32)$$

Using the non-linear Feynman-Kac formula (see [17, 18]), we argue that under suitable smoothness conditions imposed on the coefficients  $\mu, \sigma, \kappa$  and  $\beta$ , the *hedger's pricing function*  $v(t, s) := Y_t^{h,x_1,s}$  belongs to the class  $C^{1,2}([0, T] \times \mathcal{O})$  and solves the following *pricing PDE*

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \mathcal{L}v(t, s) = \bar{g}^h(t, x_1, s, v(t, s), \sigma(t, s) \frac{\partial v}{\partial s}), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}, \end{cases} \quad (5.33)$$

where the differential operator  $\mathcal{L}$  is given by the following expression

$$\mathcal{L} := \frac{1}{2} \sigma^2(t, s) \frac{\partial^2}{\partial s^2} + (\beta - \kappa)(t, s) \frac{\partial}{\partial s}.$$

In view of the definition of  $\bar{g}^h$ , it is clear that PDE (5.33) is in turn equivalent to

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2}\sigma^2(t, s)\frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s)\frac{\partial v}{\partial s}(t, s) - x_1 r_t^l B_t^l \mathbf{1}_{\{x_1 \geq 0\}} - x_1 r_t^b B_t^b \mathbf{1}_{\{x_1 \leq 0\}} - r_t^{1,b} \left( s \frac{\partial v}{\partial s}(t, s) \right)^+ \\ \quad + r_t^l \left( v(t, s) + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} + \left( s \frac{\partial v}{\partial s}(t, s) \right)^- \right)^+ \\ \quad - r_t^b \left( v(t, s) + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} + \left( s \frac{\partial v}{\partial s}(t, s) \right)^- \right)^-, \quad (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), \quad s \in \mathcal{O}. \end{cases} \quad (5.34)$$

**Remark 5.6** It is worth stressing that the coefficient  $\beta$  does not appear in the pricing PDE (5.34). Therefore, in order to derive the PDE,  $\beta$  can be chosen arbitrarily, except for constraint ensuring that the model is arbitrage-free (see Proposition 3.2). Consequently, without changing the probability measure (i.e., by choosing  $\beta$  such  $a_t = 0$  for all  $t \in [0, T]$ ), we can still derive PDE (5.34).

Conversely, if  $v \in C^{1,2}([0, T] \times \mathcal{O})$  solves PDE (5.34), then the pair  $(v(u, S_u), \sigma(u, S_u) \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (5.32) on  $u \in [t, T]$  where, for brevity, we write  $S = S^{s,t}$ . From the above discussions,  $(v(u, S_u), \frac{\partial v}{\partial s}(u, S_u))$  is also a solution to BSDE (5.31) on  $u \in [t, T]$  for an arbitrary initial stock price  $S_t = s$ . Consequently, the unique replicating strategy for the hedger equals  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  where, for all  $u \in [t, T]$ ,

$$\begin{aligned} \xi_u &= \frac{\partial v}{\partial s}(u, S_u), \quad \psi_t^{1,b} = -(B_u^{1,b})^{-1} (S_u \frac{\partial v}{\partial s}(u, S_u))^+, \\ \psi_u^l &= (B_u^l)^{-1} \left( v(u, S_u) + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} + \left( S_u \frac{\partial v}{\partial s}(u, S_u) \right)^- \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( v(u, S_u) + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} + \left( S_u \frac{\partial v}{\partial s}(u, S_u) \right)^- \right)^-. \end{aligned} \quad (5.35)$$

Let us now focus on the pricing PDE for the counterparty. Recall that  $P^c(x_2, -A, -C) = \tilde{Y}^{c,x_2}$ , where  $(\tilde{Y}^{c,x_2}, \tilde{Z}^{c,x_2})$  is the unique solution to the following BSDE

$$\begin{cases} d\tilde{Y}_u^{c,x_2} = \tilde{Z}_u^{c,x_2} \sigma(u, S_u) d\tilde{W}_u + g^c(u, x_2, S_u, \tilde{Y}_u^{c,x_2}, \tilde{Z}_u^{c,x_2}) du, \\ \tilde{Y}_T^{c,x_2} = H(S_T^{s,t}), \end{cases} \quad (5.36)$$

where for  $x_1 \geq 0$ ,

$$g^c(u, x_2, s, y, z) := z\beta(u, s) + x_2 r_u^l B_u^l + r_u^{1,b} (zs)^+ - r_u^l \left( -y + x_2 B_u^l + (-zs)^- \right)^+ + r_u^b \left( -y + x_2 B_u^l + (-zs)^- \right)^-$$

and for  $x_1 \leq 0$

$$g^c(u, x_2, s, y, z) := z\beta(u, s) + x_2 r_u^b B_u^b + r_u^{1,b} (zs)^+ - r_u^l \left( -y + x_2 B_u^b + (-zs)^- \right)^+ + r_u^b \left( -y + x_2 B_u^b + (-zs)^- \right)^-.$$

The unique replicating strategy for the counterparty equals  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  where  $\xi_u = -\tilde{Z}_u^{c,x_2}$ ,  $\psi_u^{1,b} = -(B_u^{1,b})^{-1} (\xi_u S_u)^+$  and

$$\begin{aligned} \psi_u^l &= (B_u^l)^{-1} \left( -\tilde{Y}_u^{h,x_2} + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + (\xi_u S_u)^- \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( -\tilde{Y}_u^{h,x_2} + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + (\xi_u S_u)^- \right)^-. \end{aligned}$$

For a fixed  $(t, s) \in [0, T] \times \mathcal{O}$ , we denote  $(Y_u^{c,x_2,s}, Z_u^{c,x_2,s}) := (\tilde{Y}_u^{c,x_2,s}, \tilde{Z}_u^{c,x_2,s} \sigma(u, S_u^{s,t}))$  and

$$\bar{g}^c(u, x_2, s, y, z) = g^c(u, x_2, s, y, z\sigma^{-1}(u, s)).$$



Then BSDE (5.36) becomes

$$\begin{cases} dY_u^{c,x_2,s} = Z_u^{c,x_2,s} d\widetilde{W}_u + \overline{g}^c(u, x_2, S_u^{s,t}, Y_u^{c,x_2,s}, Z_u^{c,x_2,s}) du, \\ Y_T^{c,x_2,s} = H(S_T^{s,t}). \end{cases} \quad (5.37)$$

Using the same argument as for the hedger, we deduce that the pricing function  $v(t, s) := Y_t^{c,x_2,s}$  belongs to  $C^{1,2}([0, T] \times \mathcal{O})$  and solves the following PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \mathcal{L}v(t, s) = \overline{g}^c(t, x_2, s, v(t, s), \sigma(t, s) \frac{\partial v}{\partial s}), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}, \end{cases} \quad (5.38)$$

or, more explicitly,

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2}\sigma^2(t, s) \frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s) \frac{\partial v}{\partial s}(t, s) + x_2 r_t^l B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 r_t^b B_t^b \mathbf{1}_{\{x_2 \leq 0\}} + r_t^{1,b} (s \frac{\partial v}{\partial s}(t, s))^+ \\ \quad - r_t^l \left( -v(t, s) + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} + \left( -s \frac{\partial v}{\partial s}(t, s) \right)^- \right)^+ \\ \quad + r_t^b \left( -v(t, s) + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} + \left( -s \frac{\partial v}{\partial s}(t, s) \right)^- \right)^-, & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases} \quad (5.39)$$

Conversely, if a function  $v \in C^{1,2}([0, T] \times \mathcal{O})$  solves PDE (5.39), then  $(v(u, S_u), \sigma(u, S_u) \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (5.37) on  $u \in [t, T]$  where we write  $S = S^{s,t}$ . Consequently, the pair  $(v(u, S_u), \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (5.36). Consequently, the unique replicating strategy for the hedger equals  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  where, for every  $u \in [t, T]$ ,

$$\begin{aligned} \xi_u &= -\frac{\partial v}{\partial s}(u, S_u), \quad \psi_u^{1,b} = -(B_u^{1,b})^{-1}(-S_u \frac{\partial v}{\partial s}(u, S_u))^+, \\ \psi_u^l &= (B_u^l)^{-1} \left( -v(u, S_u) + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + \left( -S_u \frac{\partial v}{\partial s}(u, S_u) \right)^- \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( -v(u, S_u) + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + \left( -S_u \frac{\partial v}{\partial s}(u, S_u) \right)^- \right)^-. \end{aligned} \quad (5.40)$$

In summary, we are in a position to formulate the following proposition.

**Proposition 5.10** *Let  $v(t, s) \in C^{1,2}([0, T] \times \mathcal{O})$  be the solution of quasi-linear PDE (5.34). Then the hedger's ex-dividend price of the European contingent claim  $H(S_T)$  is given by  $v(t, S_t)$  and the unique replicating strategy  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  for the hedger is given by (5.35). Similarly, if  $v(t, s) \in C^{1,2}([0, T] \times \mathcal{O})$  is the solution of quasi-linear PDE (5.39), then the counterparty's ex-dividend price of the European contingent claim  $H(S_T)$  is given by  $v(t, S_t)$  and the unique replicating strategy  $\varphi = (\xi, \psi^l, \psi^b, \psi^{1,b})$  for the counterparty is given by (5.40).*

If smoothness of model coefficients is not postulated then, from Theorem 4.3 in Peng [17], the function  $v(t, s) := Y_t^{h,x_1,s}$  (resp.,  $v(t, s) := Y_t^{c,x_2,s}$ ) is known to be the unique viscosity solution of PDE (5.34) (resp., (5.39)).

We notice that PDE (5.34) depends on the initial endowment  $x_1$ . In the special case where  $r^l = r^b = r$ , equation (5.34) reduces to the following PDE independent of  $x_1$

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2}\sigma^2(t, s) \frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s) \frac{\partial v}{\partial s}(t, s) - r_t^{1,b} (s \frac{\partial v}{\partial s}(t, s))^+ \\ \quad + r_t \left( v(t, s) + (s \frac{\partial v}{\partial s}(t, s))^+ \right), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases} \quad (5.41)$$

Note that PDE (5.41) can characterize the price and the strategy for the European contingent claim in the case where the borrowing rate and the lending rates are equal.

If we assume, in addition, that  $r^{i,b} = r$ , then PDE (5.41) becomes

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2}\sigma^2(t, s)\frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s)\frac{\partial v}{\partial s}(t, s) \\ \quad + r_t\left(v(t, s) - s\frac{\partial v}{\partial s}(t, s)\right), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases} \quad (5.42)$$

We observe that PDE (5.42) is nothing else but the classic Black and Scholes PDE. We mentioned in Example 3.1 that the market model partial netting does not cover the standard case of different borrowing and lending rates when  $r^{i,b} = r^b > r^l$  and trading is assumed to be unrestricted. However, when the equalities  $r^{i,b} = r^b = r^l$  are postulated, then the related PDEs for the European contingent claim are identical so, as expected, the prices and hedging strategies coincide as well.

Without using the BSDEs method, one can still obtain Proposition 5.10 by applying the classical arguments, as was done, for instance, in [1]. Both methods essentially hinge on the same tool, the non-linear Feynman-Kac formula. We mention that when the solution of the related PDE is not smooth, then the BSDE approach gives a probabilistic representation for the viscosity solution of the PDE.

**Remark 5.7** In the related paper [15], we also revisit the market model studied by Bergman [1] and we extend his analysis by considering a general contract  $(A, C)$ , rather than path-independent European claims, and investors with non-zero initial endowments. In this model, the funding accounts for risky assets are not introduced and thus the last constraint in (2.8) is relaxed. Hence the hedger can use his initial endowment to buy shares for the purpose of hedging. Consequently, for each particular set-up, the properties of prices will be quite different, but most of them can be deduced from the general results for the auxiliary BSDEs. We also derive the pricing PDE for path-independent European claims in a Markovian framework.

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## 6 Appendix: Proofs of Propositions 5.2 and 5.4

### 6.1 Proof of Proposition 5.2

*Proof of Proposition 5.2.* We assume that  $x_1 \leq 0$  and  $x_2 \leq 0$ . From Propositions 4.1 and 4.2, we know that  $P^h(x_1, A, C) = B^b(Y^{h,b,x_1} - x_1) - C$  where  $(Y^{h,b,x_1}, Z^{h,b,x_1})$  is the unique solution of BSDE (4.8) and  $P^c(x_2, -A, -C) = -(B^b(Y^{c,b,x_2} - x_2) + C)$  where  $(Y^{c,b,x_2}, Z^{c,b,x_2})$  is the unique solution of BSDE (4.12). As in the proof of Proposition 5.1, to establish (5.2), it suffices to check that  $\bar{Y}^{c,b,x_2} \leq \bar{Y}^{h,b,x_1}$  where  $(\bar{Y}^{h,b,x_1}, \bar{Z}^{h,b,x_1})$  is the unique solution to the following BSDE

$$\begin{cases} d\bar{Y}_t^{h,b,x_1} = \bar{Z}_t^{h,b,x_1,*} d\tilde{S}_t^{b,\text{cld}} + \tilde{f}_b(t, \bar{Y}_t^{h,b,x_1} + x_1, \bar{Z}_t^{h,b,x_1}) dt + (B_t^b)^{-1} dA_t^C, \\ \bar{Y}_T^{h,b,x_1} = 0, \end{cases}$$

and  $(\bar{Y}^{c,b,x_2}, \bar{Z}^{c,b,x_2})$  is the unique solution to the BSDE

$$\begin{cases} d\bar{Y}_t^{c,b,x_2} = \bar{Z}_t^{c,b,x_2,*} d\tilde{S}_t^{b,\text{cld}} - \tilde{f}_b(t, -\bar{Y}_t^{c,b,x_2} + x_2, -\bar{Z}_t^{c,b,x_2}) dt + (B_t^b)^{-1} dA_t^C, \\ \bar{Y}_T^{c,b,x_2} = 0. \end{cases}$$

To apply Theorem 3.3 in [14], we need to prove that either

$$-\tilde{f}_b(t, \bar{Y}_t^{h,b,x_1} + x_1, \bar{Z}_t^{h,b,x_1}) \geq \tilde{f}_b(t, -\bar{Y}_t^{c,b,x_2} + x_2, -\bar{Z}_t^{c,b,x_2}), \quad \mathbb{P}^b \otimes \ell - \text{a.e.}$$

or

$$-\tilde{f}_b(t, \bar{Y}_t^{c,b,x_2} + x_1, \bar{Z}_t^{c,b,x_2}) \geq \tilde{f}_b(t, -\bar{Y}_t^{c,b,x_2} + x_2, -\bar{Z}_t^{c,b,x_2}), \quad \tilde{\mathbb{P}}^b \otimes \ell - \text{a.e.}$$

To establish these inequalities, it suffices to use Lemma 6.1. We conclude that inequality (5.2) is valid.  $\square$

**Lemma 6.1** *Assume that  $x_1 \leq 0$  and  $x_2 \leq 0$ . Then the mapping  $\tilde{f}_b : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  given by equation (2.17) satisfies*

$$-\tilde{f}_b(t, y + x_1, z) \geq \tilde{f}_b(t, -y + x_2, -z), \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \tilde{\mathbb{P}}^b \otimes \ell - \text{a.e.} \quad (6.1)$$

*Proof.* Recall that, for all  $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ ,

$$\tilde{f}_b(t, y, z) := (B_t^b)^{-1} f_b(t, B_t^b y, z) - r_t^b y$$

where

$$f_b(t, y, z) := \sum_{i=1}^d r_t^b z^i S_t^i - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (\hat{z}_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (\hat{z}_t^i)^- \right)^-.$$

We now denote  $\hat{z}_t^i = (B_t^b)^{-1} z^i S_t^i$ . Then

$$\begin{aligned} \delta &:= \tilde{f}_b(t, y + x_1, z) + \tilde{f}_b(t, -y + x_2, -z) \\ &= -r_t^b(y + x_1) + (B_t^b)^{-1} f_b(t, B_t^b(y + x_1), z) - r_t^b(-y + x_2) + (B_t^b)^{-1} f_b(t, B_t^b(-y + x_2), -z) \\ &= -r_t^b(x_1 + x_2) - \sum_{i=1}^d r_t^{i,b} |\hat{z}_t^i| + r_t^l(\delta_1^+ + \delta_2^+) - r_t^b(\delta_1^- + \delta_2^-) \end{aligned}$$

where

$$\delta_1 := y + x_1 + \sum_{i=1}^d (\hat{z}_t^i)^-, \quad \delta_2 := -y + x_2 + \sum_{i=1}^d (-\hat{z}_t^i)^-.$$

Since  $r^l \leq r^b$ , we have

$$\begin{aligned} \delta &= -r_t^b(x_1 + x_2) - \sum_{i=1}^d r_t^{i,b} |\hat{z}_t^i| + r_t^l(\delta_1^+ + \delta_2^+) - r_t^b(\delta_1^- + \delta_2^-) \\ &\leq -r_t^b(x_1 + x_2) - \sum_{i=1}^d r_t^{i,b} |\hat{z}_t^i| + r_t^b(\delta_1 + \delta_2) \\ &= -r_t^b(x_1 + x_2) - \sum_{i=1}^d r_t^{i,b} |\hat{z}_t^i| + r_t^b(x_1 + x_2 + \sum_{i=1}^d |\hat{z}_t^i|) \\ &= \sum_{i=1}^d (r_t^b - r_t^{i,b}) |\hat{z}_t^i| \leq 0. \end{aligned}$$

Thus inequality (6.1) holds.  $\square$

## 6.2 Proof of Proposition 5.4

*Proof of part (i) in Proposition 5.4.* We first prove that if the initial endowments satisfy  $x_1 x_2 = 0$ , then the inequality  $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$  holds for any contract  $(A, C)$ . From Proposition 5.3, if  $x_1 \geq 0, x_2 \leq 0$ , then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}$  we have  $P^h(x_1, A, C) = \tilde{Y}^{h,l,x_1} - C$  and  $P^c(x_2, -A, -C) = \tilde{Y}^{c,b,x_2} - C$  where  $(\tilde{Y}^{h,l,x_1}, \tilde{Z}^{h,l,x_1})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{h,l,x_1} = \tilde{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{\text{cld}} + g^{h,l}(t, x_1, \tilde{Y}_t^{h,l,x_1}, \tilde{Z}_t^{h,l,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,l,x_1} = 0, \end{cases} \quad (6.2)$$

and  $(\tilde{Y}^{c,b,x_2}, \tilde{Z}^{c,b,x_2})$  is the unique solution of the following BSDE

$$\begin{cases} d\tilde{Y}_t^{c,b,x_2} = \tilde{Z}_t^{c,b,x_2,*} d\tilde{S}_t^{\text{cld}} + g^{c,b}(t, x_2, \tilde{Y}_t^{c,b,x_2}, \tilde{Z}_t^{c,b,x_2}) dt + dA_t^C, \\ \tilde{Y}_T^{c,b,x_2} = 0, \end{cases} \quad (6.3)$$

where the generators are given by

$$g^{h,l}(t, x, y, z) := \sum_{i=1}^d \beta_t^i z^i S_t^i - x r_t^l B_t^l + g(t, y + x B_t^l, z)$$

and

$$g^{c,b}(t, x, y, z) := \sum_{i=1}^d \beta_t^i z^i S_t^i + x r_t^b B_t^b - g(t, -y + x B_t^b, -z)$$

where in turn (see (5.3))

$$g(t, y, z) = -\sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left( y + \sum_{i=1}^d (z^i S_t^i)^- \right)^-. \quad (6.4)$$

Let us denote  $\bar{z}_t^i = z^i S_t^i$ . To apply Theorem 3.3 in [14], it suffices to show that

$$-\sum_{i=1}^d \beta_t^i \bar{z}_t^i + x_1 r_t^l B_t^l - g(t, y + x_1 B_t^l, z) \geq -\sum_{i=1}^d \beta_t^i \bar{z}_t^i - x_2 r_t^b B_t^b + g(t, -y + x_2 B_t^b, -z),$$

which is equivalent to

$$\delta := g(t, y + x_1 B_t^l, z) + g(t, -y + x_2 B_t^b, -z) - x_1 r_t^l B_t^l - x_2 r_t^b B_t^b \leq 0.$$

In view of Lemma 6.2, we conclude that the inequality  $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$  holds for every  $t \in [0, T]$ .  $\square$

*Proof of part (ii) in Proposition 5.4.* We now assume that  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . Assume that  $x_1 x_2 \neq 0$ , that is,  $x_1 > 0$  and  $x_2 < 0$ . Our goal is to find a contract  $(A, C)$  and a date  $\hat{t} \in [0, T]$  such that the inequality  $P_{\hat{t}}^c(x_2, -A, -C) \leq P_{\hat{t}}^h(x_1, A, C)$  fails to hold. To this end, we consider a contract with  $C = 0$  and

$$A_t = p \mathbf{1}_{[0, T]}(t) - \alpha \mathbf{1}_{[t_0, T]}(t) + \alpha e^{\int_{t_0}^T r_u du} \mathbf{1}_{[T]}(t),$$

where  $t_0 \in (0, T)$  and the function  $r$  satisfies  $r_u \in (r_u^l, r_u^b)$  for all  $u \in [0, T]$ . Moreover, a constant  $\alpha > 0$  is such that

$$\begin{aligned} x_1 B_{t_0}^l - \alpha e^{\int_{t_0}^T (r_u - r_u^l) du} &\geq 0, & x_1 + \alpha (B_{t_0}^l)^{-1} - \alpha (B_T^l)^{-1} e^{\int_{t_0}^T r_u du} &\geq 0, \\ x_2 B_{t_0}^b + \alpha e^{\int_{t_0}^T (r_u - r_u^b) du} &\leq 0, & x_2 - \alpha (B_{t_0}^b)^{-1} + \alpha (B_T^b)^{-1} e^{\int_{t_0}^T r_u du} &\leq 0, \end{aligned}$$

which in turn is equivalent to:  $\alpha > 0$  and

$$x_2 \kappa_2^{-1} \leq \alpha \leq \min \left\{ x_1 B_{t_0}^l e^{-\int_{t_0}^T (r_u - r_u^l) du}, -x_2 B_{t_0}^b e^{-\int_{t_0}^T (r_u - r_u^b) du}, x_1 \kappa_1^{-1} \right\} \quad (6.5)$$

where

$$\kappa_1 := -(B_{t_0}^l)^{-1} + (B_T^l)^{-1} e^{\int_{t_0}^T r_u du}, \quad \kappa_2 := (B_{t_0}^b)^{-1} + (B_T^b)^{-1} e^{\int_{t_0}^T r_u du}.$$

Note that  $\kappa_1 > 0$  and  $\kappa_2 > 0$  since from  $r^l < r < r^b$ , we obtain

$$-\int_0^{t_0} r_u^l du - \left( \int_{t_0}^T r_u du - \int_0^T r_u^l du \right) = -\int_{t_0}^T (r_u - r_u^l) du < 0.$$

and

$$-\int_0^{t_0} r_u^b du - \left( \int_{t_0}^T r_u du - \int_0^T r_u^b du \right) = -\int_{t_0}^T (r_u - r_u^b) du > 0,$$

Therefore, a constant  $\alpha > 0$  satisfying (6.5) exists and for

$$x := x_1 + \alpha (B_{t_0}^l)^{-1} - \alpha (B_T^l)^{-1} e^{\int_{t_0}^T r_u du} \geq 0,$$

we obtain

$$x B_{t_0}^l - \alpha = x_1 B_{t_0}^l - \alpha e^{\int_{t_0}^T (r_u - r_u^l) du} \geq 0.$$

Now we define the strategy  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{d,b}, \eta^b, \eta^l)$  where  $\xi^i = \psi^{i,b} = \psi^b = \eta^b = \eta^l = 0$  for  $i = 1, 2, \dots, d$  and

$$\psi_t^l = x \mathbf{1}_{[0, t_0)} + (B_{t_0}^l)^{-1} (x B_{t_0}^l - \alpha) \mathbf{1}_{[t_0, T)} + (B_T^l)^{-1} \left( x B_T^l - \alpha e^{\int_{t_0}^T r_u^l du} + \alpha e^{\int_{t_0}^T r_u du} \right) \mathbf{1}_{[T, T]}.$$

The wealth process satisfies

$$\begin{aligned} V_T(x, \varphi, A, C) &= (x B_{t_0}^l - \alpha) e^{\int_{t_0}^T r_u^l du} + \alpha e^{\int_{t_0}^T r_u du} = x B_T^l - \alpha e^{\int_{t_0}^T r_u^l du} + \alpha e^{\int_{t_0}^T r_u du} \\ &= \left( x_1 + \alpha (B_{t_0}^l)^{-1} - \alpha (B_T^l)^{-1} e^{\int_{t_0}^T r_u du} \right) B_T^l - \alpha e^{\int_{t_0}^T r_u^l du} + \alpha e^{\int_{t_0}^T r_u du} \\ &= x_1 B_T^l = V_T^0(x_1). \end{aligned}$$

Therefore, the self-financing strategy  $(x, \varphi, A, C)$  replicates the contract  $(A, C)$  on  $[0, T]$ . Moreover from the uniqueness of the related pricing BSDE, we know that this is the unique strategy. From Definition 3.7, it follows that

$$P_0^h(x_1, A, C) = x - x_1 = \alpha (B_{t_0}^l)^{-1} - \alpha (B_T^l)^{-1} e^{\int_{t_0}^T r_u du} = -\alpha \kappa_1 < 0.$$

Let us now focus on the counterparty. If we set

$$\tilde{x} = x_2 - \alpha (B_{t_0}^b)^{-1} + \alpha (B_T^b)^{-1} e^{\int_{t_0}^T r_u du} \leq 0,$$

then we obtain

$$\tilde{x} B_{t_0}^b + \alpha = x_2 B_{t_0}^b + \alpha e^{\int_{t_0}^T (r_u - r_u^b) du} \leq 0.$$

We define the strategy  $\tilde{\varphi} = (\tilde{\xi}^1, \dots, \tilde{\xi}^d, \tilde{\psi}^l, \tilde{\psi}^b, \tilde{\psi}^{1,b}, \dots, \tilde{\psi}^{d,b}, \tilde{\eta}^b, \tilde{\eta}^l)$  where  $\tilde{\xi}^i = \tilde{\psi}^{i,b} = \tilde{\psi}^l = \tilde{\eta}^b = \tilde{\eta}^l = 0$  for  $i = 1, 2, \dots, d$  and

$$\tilde{\psi}_t^b = \tilde{x} \mathbf{1}_{[0, t_0)} + (B_{t_0}^b)^{-1} (\tilde{x} B_{t_0}^b + \alpha) \mathbf{1}_{[t_0, T)} + (B_T^b)^{-1} \left( \tilde{x} B_T^b + \alpha e^{\int_{t_0}^T r_u^b du} - \alpha e^{\int_{t_0}^T r_u du} \right) \mathbf{1}_{[T, T]}.$$

Then we have

$$\begin{aligned} V_T(\tilde{x}, \tilde{\varphi}, -A, -C) &= (\tilde{x} B_{t_0}^b + \alpha) e^{\int_{t_0}^T r_u^b du} - \alpha e^{\int_{t_0}^T r_u du} = \tilde{x} B_T^b + \alpha e^{\int_{t_0}^T r_u^b du} - \alpha e^{\int_{t_0}^T r_u du} \\ &= \left( x_2 - \alpha (B_{t_0}^b)^{-1} + \alpha (B_T^b)^{-1} e^{\int_{t_0}^T r_u du} \right) B_T^b + \alpha e^{\int_{t_0}^T r_u^b du} - \alpha e^{\int_{t_0}^T r_u du} \\ &= x_2 B_T^b = V_T^0(x_2). \end{aligned}$$

Hence  $(\tilde{x}, \tilde{\varphi}, -A, -C)$  is the unique self-financing strategy replicating the contract  $(-A, -C)$  on  $[0, T]$ . From Definition 3.8, it follows that

$$P_0^c(x_2, -A, -C) = x_2 - \tilde{x} = \alpha (B_{t_0}^b)^{-1} - \alpha (B_T^b)^{-1} e^{\int_{t_0}^T r_u du} = \alpha \kappa_2 > 0.$$

We have thus found a date  $\hat{t} = 0$  and a contract  $(A, C)$  such that

$$P_0^c(x_2, -A, -C) > \alpha \kappa_2 > 0 > -\alpha \kappa_1 = P_0^h(x_1, A, C),$$

so that the range of bilaterally profitable prices  $\mathcal{R}_0^p(x_1, x_2)$  is non-empty. Note that here both parties are willing to pay a strictly positive amount to the other party for the right to enter the contract. This completes the proof of part (ii).  $\square$

**Lemma 6.2** *Assume that  $x_1 x_2 = 0$ . If the function  $g$  is given by (6.4), then following inequality holds*

$$\delta := g(t, y + x_1 B_t^l, z) + g(t, -y + x_2 B_t^b, -z) - x_1 r_t^l B_t^l - x_2 r_t^b B_t^b \leq 0.$$

*Proof.* In view of (6.4), we have

$$\delta = -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b - \sum_{i=1}^d r_t^{i,b} |\bar{z}_t^i| + r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-),$$

where

$$\delta_1 = y + x_1 B_t^l + \sum_{i=1}^d (\bar{z}_t^i)^-, \quad \delta_2 = -y + x_2 B_t^b + \sum_{i=1}^d (-\bar{z}_t^i)^-.$$

We claim that

$$\delta \leq \min \left\{ (r_t^l - r_t^b) x_2 B_t^b + \sum_{i=1}^d (r_t^l - r_t^{i,b}) |\bar{z}_t^i|, (r_t^b - r_t^l) x_1 B_t^l + \sum_{i=1}^d (r_t^b - r_t^{i,b}) |\bar{z}_t^i| \right\}.$$

Indeed, from  $r^l \leq r^b$ , we have

$$\begin{aligned} & r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) \\ & \leq \min \{ r_t^l (\delta_1 + \delta_2), r_t^b (\delta_1^- + \delta_2^-) \} \\ & = \min \left\{ r_t^l (x_1 B_t^l + x_2 B_t^b + \sum_{i=1}^d |\bar{z}_t^i|), r_t^b (x_1 B_t^l + x_2 B_t^b + \sum_{i=1}^d |\bar{z}_t^i|) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \delta &= -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b - \sum_{i=1}^d r_t^{i,b} |\bar{z}_t^i| + r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) \\ &\leq -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b - \sum_{i=1}^d r_t^{i,b} |\bar{z}_t^i| \\ &\quad + \min \left\{ r_t^l (x_1 B_t^l + x_2 B_t^b + \sum_{i=1}^d |\bar{z}_t^i|), r_t^b (x_1 B_t^l + x_2 B_t^b + \sum_{i=1}^d |\bar{z}_t^i|) \right\} \\ &= \min \left\{ (r_t^l - r_t^b) x_2 B_t^b + \sum_{i=1}^d (r_t^l - r_t^{i,b}) |\bar{z}_t^i|, (r_t^b - r_t^l) x_1 B_t^l + \sum_{i=1}^d (r_t^b - r_t^{i,b}) |\bar{z}_t^i| \right\}. \end{aligned}$$

If  $x_1 x_2 = 0$ , then the right-hand side in the above inequality is non-positive. We conclude that  $\delta \leq 0$ .  $\square$